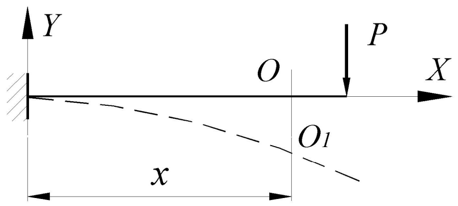
ÏÅÐÅÌÅÙÅÍÈß ÏÐÈ ÈÇÃÈÁÅ

1. Ïðîãèá è óãîë ïîâîðîòà ñå÷åíèÿ áàëêè

Ïðèëîæèì ê êîíñîëüíîé áàëêå âíåøíþþ ñèëó Ð (Ðèñ.1)



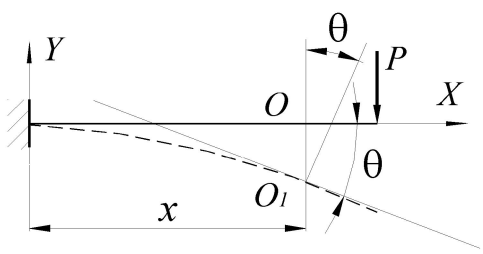
Ðèñ 1

Èçîáðàçèì èñêðèâëåííóþ îñü áàëêè, êîòîðàÿ ïîëó÷èòñÿ îò âîçäåéñòâèÿ ýòîé ñèëû.

Ïðè ýòîì öåíòð òÿæåñòè Î êàêîãî–ëèáî ñå÷åíèÿ ñ àáñöèññîé õ ïåðåìåùàåòñÿ â òî÷êó O_1.

Ïåðåìåùåíèå ÎO_1 öåíòðà òÿæåñòè ñå÷åíèÿ ïî íàïðàâëåíèþ, ïåðïåíäèêóëÿðíîìó îñè áàëêè, íàçûâàåòñÿ ïðîãèáîì áàëêè â ýòîì ñå÷åíèè èëè ïðîãèáîì ýòîãî ñå÷åíèÿ áàëêè. Ïðîãèá ïðèíÿòî îáîçíà÷àòü ëàòèíñêîé áóêâîé ó.

Êðîìå òîãî, ïðè äåôîðìàöèè áàëêè ñå÷åíèå, îñòàâàÿñü ïëîñêèì, ïîâîðà÷èâàåòñÿ ïî îòíîøåíèþ ê ïåðâîíà÷àëüíîìó ïîëîæåíèþ (Ðèñ.2).



Ðèñ 2

Óãîë, íà êîòîðûé êàæäîå ñå÷åíèå ïîâîðà÷èâàåòñÿ ïî îòíîøåíèþ ê ñâîåìó ïåðâîíà÷àëüíîìó ïîëîæåíèþ, íàçûâàåòñÿ óãëîì ïîâîðîòà ñå÷åíèÿ. Óãîë ïîâîðîòà ïðèíÿòî îáîçíà÷àòü ãðå÷åñêîé áóêâîé Q.

Íà ïðèâåäåííîì âûøå ÷åðòåæå âûáåðåì ñèñòåìó êîîðäèíàò. Íà÷àëî êîîðäèíàò ðàñïîëîæèì â çàäåëêå. Îñü Y íàïðàâèì ââåðõ, à îñü Õ– âïðàâî. Òîãäà óðàâíåíèå ó = f(x) áóäåò ïðåäñòàâëÿòü ñîáîé óðàâíåíèå êðèâîé, ïî êîòîðîé èçîãíåòñÿ îñü áàëêè îò äåéñòâèÿ âíåøíèõ ñèë, òî åñòü ýòî áóäåò óðàâíåíèå èçîãíóòîé îñè áàëêè.

Â òî÷êó O_1 ïðîâåäåì êàñàòåëüíóþ, êîòîðàÿ ñîñòàâèò ñ îñüþ Õ óãîë, ðàâíûé óãëó ïîâîðîòà ïîïåðå÷íîãî ñå÷åíèÿ Q. Âìåñòå ñ òåì ïî ãåîìåòðè÷åñêîìó ñìûñëó ïðîèçâîäíîé òàíãåíñ óãëà êàñàòåëüíîé ê êðèâîé ó = f(x) ñ îñüþ Õ åñòü: tgQ = dy/dx.

Èç–çà ìàëîñòè äåôîðìàöèé òàíãåíñ óãëà ïîâîðîòà ìîæíî ïðèðàâíÿòü ñàìîìó óãëó. Òîãäà ìîæåì çàïèñàòü:

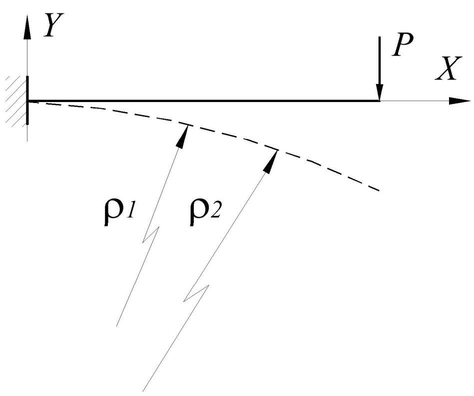
$$Q = dy/dx. \qquad (1)$$

Èç ôîðìóëû ñëåäóåò, ÷òî óãîë ïîâîðîòà ñå÷åíèÿ ðàâåí ïåðâîé ïðîèçâîäíîé ïî àáñöèññå ñå÷åíèÿ õ îò ïðîãèáà ó â íåì.

Ñëåäîâàòåëüíî, çàäà÷à èçó÷åíèÿ äåôîðìàöèè áàëêè ñâîäèòñÿ ê ïîëó÷åíèþ óðàâíåíèÿ èçîãíóòîé îñè ó = f(õ).

7.2. Äèôôåðåíöèàëüíîå óðàâíåíèå èçîãíóòîé îñè áàëêè

Èçîáðàçèì ðàäèóñû êðèâèçíû, ïðîâåäåííûå ê èçîãíóòîé îñè áàëêè



Ðèñ 3

Äëÿ ïîëó÷åíèÿ óðàâíåíèÿ èçîãíóòîé îñè èñïîëüçóåì ìàòåìàòè÷åñêóþ çàâèñèìîñòü ìåæäó ðàäèóñîì êðèâèçíû èçîãíóòîé îñè è êîîðäèíàòàìè åãî òî÷åê õ è ó:

$1/r = \pm (d^2y/dx^2)/[1 + (dy/dx)^2]^{3/2}.$

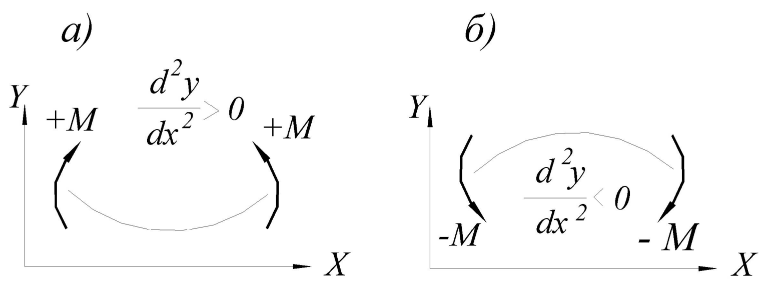
Ïîäñòàâèì ïîëó÷åííîå çíà÷åíèå $1/r$ â ôîðìóëó (6.2)

$\pm E(d^2y/dx^2)/[1 + (dy/dx)^2]^{3/2} = M/I.$

Â ïîñëåäíåì âûðàæåíèè $dx/dy = Q$, ïðåäñòàâëÿåò ñîáîé âåñüìà ìàëóþ âåëè÷èíó óãëà ïîâîðîòà ñå÷åíèÿ, äà åùå âîçâåäåííóþ â êâàäðàò, ïîýòîìó â ïðàêòè÷åñêèõ ðàñ÷åòàõ åé ìîæíî ïðåíåáðå÷ü. Òîãäà ìîæåì çàïèñàòü: $\pm E(d^2y/dx^2) = M/I$ èëè

$$\pm EI(d^2y/dx^2) = M. \qquad (à)$$

Ðàíåå óñòàíîâëåííîå ïðàâèëî çíàêîâ äëÿ èçãèáàþùèõ ìîìåíòîâ ÿâëÿåòñÿ íåçàâèñèìûì îò êîîðäèíàòíûõ îñåé, íî âòîðàÿ ïðîèçâîäíàÿ, êàê èçâåñòíî, çàâèñèò îò èõ íàïðàâëåíèÿ Îíà áóäåò ïîëîæèòåëüíîé, åñëè â ñòîðîíó ïîëîæèòåëüíîé îñè Y îáðàùåíà âîãíóòîñòü êðèâîé (Ðèñ 4 à), è îòðèöàòåëüíà– åñëè âûïóêëîñòü (Ðèñ 4 á).



Ðèñ 4

Òàêèì îáðàçîì, çíàê èçãèáàþùåãî ìîìåíòà íå çàâèñèò îò ðàñïîëîæåíèÿ êîîðäèíàòíûõ îñåé, à çíàê âòîðîé ïðîèçâîäíîé– çàâèñèò. Ïðè íàïðàâëåíèè îñè Y ââåðõ â âûðàæåíèè (à) ñëåäóåò ñòàâèòü çíàê "ïëþñ", à ïðè íàïðàâëåíèè âíèç– çíàê "ìèíóñ". Äëÿ óäîáñòâà ðàñ÷åòîâ óñëîâèìñÿ â äàëüíåéøåì âñåãäà îñü Y íàïðàâëÿòü ââåðõ. Òîãäà âûðàæåíèå (à) ìîæíî ïðåäñòàâèòü â ñëåäóþ-ùåì âèäå:

$$EI(d^2y/dx^2) = M, \qquad (2)$$

ãäå Å– ìîäóëü Þíãà, Ïà;
I– îñåâîé ìîìåíò èíåðöèè îòíîñèòåëüíî íåéòðàëüíîé îñè, ì^4;
ó– ïðîãèá ñå÷åíèÿ, ì;
õ– àáñöèññà äàííîãî ñå÷åíèÿ, ì;
Ì– èçãèáàþùèé ìîìåíò â ïîïåðå÷íîì ñå÷åíèè áàëêè, Íì.

Ôîðìóëà ÿâëÿåòñÿ äèôôåðåíöèàëüíûì óðàâíåíèåì èçîãíóòîé îñè áàëêè.

Äëÿ ïîëó÷åíèÿ èç ôîðìóëû óðàâíåíèÿ ïðîãèáîâ ó = f(x) íåîáõîäèìî åå ïðîèíòåãðèðîâàòü. Èíòåãðèðóÿ óðàâíåíèå èçîãíóòîé îñè â ïåðâûé ðàç, ïîëó÷èì:

$$EI\,(dy/dx) = EI\,Q = \text{òì}\,dx + \text{Ñ}.$$

Èíòåãðèðóÿ óðàâíåíèå èçîãíóòîé îñè âòîðîé ðàç, ïîëó÷èì:

$$EIy = \text{ò}dx\ \text{ò}Mdx + \text{Ñõ} + D.$$

Òîãäà ìû ìîæåì ïîëó÷èòü óðàâíåíèÿ äëÿ îïðåäåëåíèÿ óãëîâ ïîâîðîòà è ïðîãèáîâ:

$$Q = (1/EI)(\text{òì}\,dx + C); \qquad (3)$$

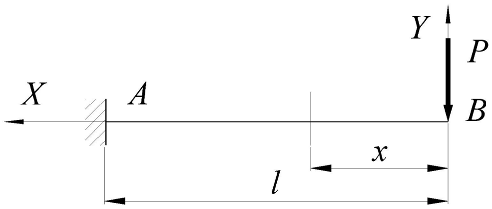
$$y = (1/EI)(\text{ò}dx\ \text{ò}Mdx + Cx + D), \qquad (4)$$

ãäå â ôîðìóëàõ Ñ è Ä– ïîñòîÿííûå èíòåãðèðîâàíèÿ, îïðåäåëÿåìûå èç ãðàíè÷íûõ óñëîâèé áàëêè.

3. Îïðåäåëåíèå äåôîðìàöèé ïðè èçãèáå ìåòîäîì íåïîñðåäñòâåííîãî èíòåãðèðîâàíèÿ äèôôåðåíöèàëüíîãî óðàâíåíèÿ èçîãíóòîé îñè

Ýòîò ìåòîä îñíîâàí íà ôîðìóëàõ (3) è (4). Ðàññìîòðèì íåêîòîðûå ïðèìåðû èõ ïðèìåíåíèÿ.

Îïðåäåëèì óãîë ïîâîðîòà è ïðîãèá ñå÷åíèÿ, ïðîõîäÿùåãî ÷åðåç òî÷êó Â áàëêè, ïðåäñòàâëåííîé íà ðèñóíêå (Ðèñ. 5).



Ðèñ 5

Âûáèðàåì íà÷àëî êîîðäèíàò â òî÷êå Â. Ïðîâîäèì ïîïåðå÷íîå ñå÷åíèå, èçãèáàþùèé ìîìåíò â êîòîðîì áóäåò ðàâåí Ì = –Ðõ. Òîãäà äèôôåðåíöèàëüíîå óðàâíåíèå èçîãíóòîé îñè áóäåò èìåòü âèä:

$$EI(d^2y/dx^2) = -Px.$$

Èíòåãðèðóåì åãî äâà ðàçà:

EIQ = –Ðx^2/2 +Ñ; (a)

EIy = –Ðx^3/6 + Ñõ +D. (b)

Ïîñòîÿííûå èíòåãðèðîâàíèÿ îïðåäåëÿåì èç òîãî óñëîâèÿ, ÷òî ïðîãèá è óãîë ïîâîðîòà â çàäåëêå (ïðè õ = l) ðàâíû íóëþ.

EIQ$_A$ (ïðè õ=1) = 0 = –Ðl^2/2 + Ñ; EIy$_A$ (ïðè õ = 1) = 0 = –Ðl^3/6 + + Cl + D, òîãäà C = Pl^2/2, a D = (Pl^3/6) – (Pl^3/2) = –Pl^3/3.

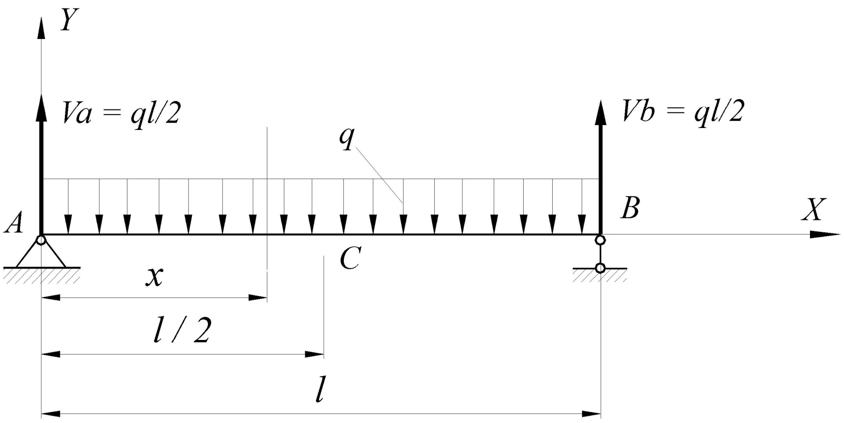
Äëÿ òîãî ÷òîáû îïðåäåëèòü ó$_Â$ è Q$_Â$, ïîäñòàâèì â óðàâíåíèÿ (à) è (b) çíà÷åíèÿ ïðîèçâîëüíûõ ïîñòîÿííûõ è õ = 0.

EIQ$_B$ = C = Pl^2/2; EIy$_B$ = D = –Pl^3/3, òîãäà Q$_B$ = Pl^2/2EI; y$_B$ = –Pl^3 /3EI.

Ðàññìîòðåííûé ïðèìåð ïîçâîëÿåò òàêæå çàêëþ÷èòü, ÷òî ïðîèçâåäåíèå æåñòêîñòè ïðè èçãèáå EI íà óãîë ïîâîðîòà è ïðîãèá â íà÷àëå ïðèíÿòîé êîîðäèíàòíîé ñèñòåìû äàåò ñîîòâåòñòâåííî ïîñòîÿííûå èíòåãðèðîâàíèÿ Ñ è D. Òî åñòü

EIQ$_0$ = EIQ$_B$ = Ñ; EIy$_0$ = EIy$_B$ = D.

Íà ïðèìåðå îäíîïðîëåòíîé áàëêè, íàãðóæåííîé ðàñïðåäåëåííîé íàãðóçêîé èíòåíñèâíîñòüþ q, îïðåäåëèì óãëû ïîâîðîòà ñå÷åíèé, ïðîâåäåííûõ ÷åðåç òî÷êè À è Â, à òàêæå ïðîãèá ñå÷åíèÿ â òî÷êå Ñ (Ðèñ 6).



Ðèñ 6

Òàê êàê âñÿ íàãðóçêà, ïðèëîæåííàÿ ê áàëêå ñèììåòðè÷íà è ðàâíà ql, òî $V_A = V_B = ql/2$. Ïðîâîäèì ïîïåðå÷íîå ñå÷åíèå, èçãèáàþùèé ìîìåíò â êîòîðîì áóäåò ðàâåí $M = (ql/2)x - (qx^2/2)$. Çàïèøåì äèôôåðåíöèàëüíîå óðàâíåíèå èçîãíóòîé îñè áàëêè è ïðîèíòåãðèðóåì åãî äâà ðàçà.

$$EI(d^2y/dx^2) = (ql/2)x - (qx^2/2); \quad EIQ = (ql/2)(x^2/2) - (qx^3/6) + C; \qquad (a)$$

$$EIy = (ql/2)(x^3/6) - (qx^4/24) + Cx + D. \qquad (b)$$

Ïîñòîÿííûå èíòåãðèðîâàíèÿ îïðåäåëèì èç óðàâíåíèÿ (b), òàê êàê ïðîãèáû â îïîðàõ ðàâíû íóëþ.

EIy_A (ïðè õ = 0) = 0 = D, òî åñòü $EIy_À = EIy_0 = D = 0$;

EIy_B (ïðè õ = 1) = 0 = $(ql/2)(l^3/6) - (ql^4/24) + Cl$, òîãäà Ñ = $-(ql^3/12)$ +

+ $ql^3/24 = -ql^3/24$.

Îïðåäåëÿåì óãëû ïîâîðîòîâ Q_A è Q_B. Ïîäñòàâëÿåì äëÿ ýòîãî â óðàâíåíèå (à) õ = 0, õ = 1 è Ñ.

EIQ_A (ïðè õ = 0) = Ñ = $-ql^3/24$. Ñëåäîâàòåëüíî, $Q_A = -ql^3/24EI$, à

$ELQ_AELQ_0 = C$.

EIQ_B (ïðè õ = 1) = 0 = $(ql/2)(l^2/2) - ql^3/6 - ql^3/24$, òîãäà Q_B =

$ql^3/24$.

Îïðåäåëÿåì ïðîãèá ó$_C$. Ïîäñòàâëÿåì äëÿ ýòîãî â óðàâíåíèå (b) õ = 1/2 è Ñ.

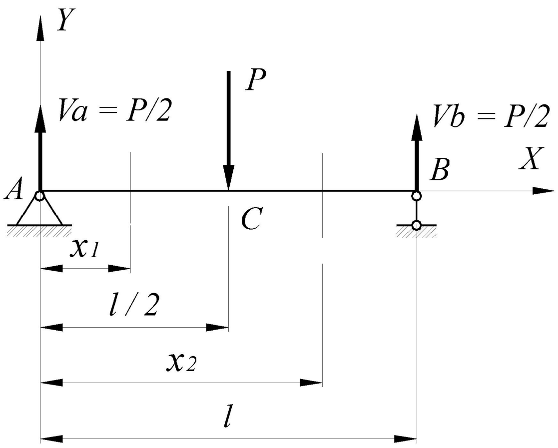
EIy$_C$ = (ïðè õ = 1) = $(ql/2)[l^3/(6\times8)] - [ql^4/(24\times16)] - (ql^3/24)(l/2)$ =

= $ql^4/96 - ql^4/384 - ql^4/48 = -5ql^4/384$, òîãäà

$$y_C = -5ql^4/384.$$

Êàê è â ïåðâîì ïðèìåðå ïîëó÷èëè, ÷òî *ïðîèçâåäåíèÿ æåñòêîñòè ïðè èçãèáå EI íà óãîë ïîâîðîòà è ïðîãèá â íà÷àëå êîîðäèíàò ðàâíû ñîîòâåòñòâåííî ïîñòîÿííûì èíòåãðèðîâàíèÿ Ñ è D.*

Äëÿ áàëêè, íàãðóæåííîé â ñåðåäèíå ïðîëåòà ñîñðåäîòî÷åííîé ñèëîé Ð, îïðåäåëèì óãëû ïîâîðîòà â ñå÷åíèÿõ, ïðîõîäÿùèõ ÷åðåç îïîðû, è ïðîãèá ñå÷åíèÿ â ñåðåäèíå ïðîëåòà Ðèñ 7.



Ðèñ 7

Òàê êàê áàëêà èìååò äâà ñèëîâûõ ó÷àñòêà, ïðîâîäèì äâà ñå÷åíèÿ ñ ïðàâîé ñòîðîíû îò íà÷àëà êîîðäèíàò.

Äëÿ êàæäîãî ñå÷åíèÿ çàïèñûâàåì äèôôåðåíöèàëüíûå óðàâíåíèÿ èçîãíóòîé îñè è èíòåãðèðóåì èõ äâà ðàçà áåç ðàñêðûòèÿ ñêîáîê.

Ïåðâîå ñå÷åíèå.

$EI(d^2y_1/dx_1^2) = (P/2)x_1$; $EIQ_1 = (P/2)x_1^2 + C_1$; (a)

$Ely_1 = (P/2)x_1^3/6 + C_1x_1 + D_1$. (b)

Âòîðîå ñå÷åíèå. $EId^2y_2/dx_2^2 = (P/2)x_2 – P(x_2–l/2)$;

$EIQ_2 = (P/2)x_2^2/2 – [P(x – l/2)^2]/2 + C_2$; (c)

$Ely_2 = (P/2)(x_2^3/6) – [P(x – l/2)^3]/6 + C_2x_2 + D_2$. (d)

Â çàïèñàííûõ íàìè óðàâíåíèÿõ èìååì ÷åòûðå ïîñòîÿííûõ èíòåãðèðîâàíèÿ. Ïîäñòàâèì â óðàâíåíèå (à) $x_1 = l/2$, à â óðàâíåíèå (ñ) $õ_2 = l/2$ è ïîëó÷èì:

EIQ_C (ïðè $õ = l/2$) $= Pl^2/16 + C_1$;

EIQ_C (ïðè $õ_2 = l/2$) $= Pl^2/16 + Ñ_2$.

Ñðàâíèâàÿ ïîëó÷åííûå âûðàæåíèÿ, ìîæåì çàêëþ÷èòü, ÷òî $C_1=Ñ_2=Ñ$.

Ïðîâåäåì ïîäîáíûå ïîäñòàíîâêè â óðàâíåíèÿ äëÿ îïðåäåëåíèÿ ïðîãèáîâ (b) è (d)

$Ely_Ñ$ (ïðè $x_1 = l/2$) $= Pl^3/96 + Ñ(l/2) + D_1$;

Ely_C (ïðè $õ_2 = l/2$) $= Pl^3/96 + C(l/2) + D_2$.

Òîãäà $D_1 = D_2 = D$. Ñëåäîâàòåëüíî, íåîáõîäèìî îïðåäåëèòü òîëüêî äâå ïîñòîÿííûõ èíòåãðèðîâàíèÿ Ñ è D.

Ïîäñòàâèì â óðàâíåíèå (b) $x_1 = 0$: Ely_A (ïðè $x_1= 0$) $= 0 = D$, èëè $Ely_A = Ely_0 = D$.

Ïîäñòàâèì â óðàâíåíèå (d)$x_2 = 1$: Ely_B (ïðè $x_2 = 1$) $= 0 = (Pl^3/12)–$
$– (Pl^3/48) + Cl$, îòñþäà $Ñ = – (Pl^2/12) + (Pl^2/48) = –Pl^2/16$.

Îïðåäåëèì óãîë ïîâîðîòà Q_A:

$ÅIQ_A$(ïðè $x_1= 0$) $= Ñ = –Pl^2/16$, òîãäà $Q_À = –Pl^2/16EI$, a $EIQ_A= EIQ_0 = C$.

Òàêèì îáðàçîì, âî âñåõ ðàññìîòðåííûõ íàìè ïðèìåðàõ ïðîèçâåäåíèå æåñòêîñòè íà óãîë ïîâîðîòà â íà÷àëå êîîðäèíàò äàåò çíà÷åíèå ïîñòîÿííîé èíòåãðèðîâàíèÿ Ñ, à ïðîèçâåäåíèå æåñòêîñòè íà ïðîãèá â íà÷àëå êîîðäèíàò – ïîñòîÿííóþ èíòåãðèðîâàíèÿ D.

$$EIQ_0 = C; \quad EIy_0 = D. \qquad (5)$$

Îïðåäåëèì $Q_Â$:

$$ÅIQ_B \ (ïðè\ x_2 = l) = (P/2)(l^2/2) – Pl^2/8 – Pl^2/16 = Pl^2/16; \quad Q_Â = Pl^2/16EI.$$

Îïðåäåëèì ïðîãèá $ó_Ñ$:

$$ÅIó_Ñ \ (ïðè\ x_1=1/2) = (P/2)(l^3/48) – (Pl^2/16)(l/2) = –Pl^3/48;$$

$$ó_Ñ = –Pl^3/48EI.$$

Ïðèìå÷àíèå: *Âî âñåõ ðàññìîòðåííûõ ïðèìåðàõ ðàçìåðíîñòüþ óãëîâ ïîâîðîòà ÿâëÿåòñÿ ðàä, ðàçìåðíîñòü ïðîãèáîâ– ì, ñì, ìì è ò.ä.*

4. Îáîáùåííîå óðàâíåíèå èçîãíóòîé îñè áàëêè èëè ìåòîä íà÷àëüíûõ ïàðàìåòðîâ

Â 1923 ãîäó ïðîôåññîð *Í.Ï. Ïóçûðåâñêèé* â òðóäå "Ðàñ÷åòû ôóíäàìåíòîâ" äàë âåñüìà óäîáíûé îáùèé ìåòîä èíòåãðèðîâàíèÿ äèôôåðåíöèàëüíîãî óðàâíåíèÿ èçîãíóòîé îñè áàëêè ïîñòîÿííîãî ñå÷åíèÿ, ëåæàùåé íà ñïëîøíîì óïðóãîì îñíîâàíèè. Ýòîò ìåòîä ïîëó÷èë íàçâàíèå "*ìåòîäà íà÷àëüíûõ ïàðàìåòðîâ".* Â ïðèìåíåíèè ê îáû÷íûì áàëêàì àíàëîãè÷íûé ìåòîä áûë ðàçðàáîòàí *Í.Ã.Êóëèêîâñêèì* â 1926 ãîäó.

Ñòðîãî ìàòåìàòè÷åñêîå îáîñíîâàíèå ýòîãî ìåòîäà è åãî äàëüíåéøåå ðàçâèòèå áûëî äàíî íåçàâèñèìî îò Í.Ï. Ïóçûðåâñêîãî âûäàþùèìñÿ ðóññêèì ó÷åíûì *À.Í. Êðûëîâûì* â êíèãå "Î ðàñ÷åòå áàëîê, ëåæàùèõ íà óïðóãîì îñíîâàíèè", èçäàííîé â 1930 ãîäó.

Àëåêñåé Íèêîëàåâè÷ Êðûëîâ (1863–1945) áûë èíæåíåðîì, ó÷åíûì– ìåõàíèêîì è ìàòåìàòèêîì. Â åãî íàó÷íîì òâîð÷åñòâå ïðåêðàñíî ñî÷åòàëèñü ïðèêëàäíûå èíòåðåñû ñ ãëóáîêèì ïîíèìàíèåì *èñõîäíûõ ïðèíöèïîâ êëàññè÷åñêîé ìåõàíèêè è ñ óäèâèòåëüíûì óìåíèåì èíòåðïðåòèðîâàòü è êîììåíòèðîâàòü èäåè åå îñíîâàòåëåé. Îäíàêî öåíòðàëüíîå ìåñòî â åãî èññëåäîâàíèÿõ çàíèìàþò ðàáîòû ïî òåîðèè êîðàáëÿ, ïðèíåñøèå åìó ìèðîâóþ ñëàâó è èçâåñòíîñòü Îñîáåííî èçâåñòíû òðóäû À.Í. Êðûëîâà î íåïîòîïëÿåìîñòè ñóäîâ.*

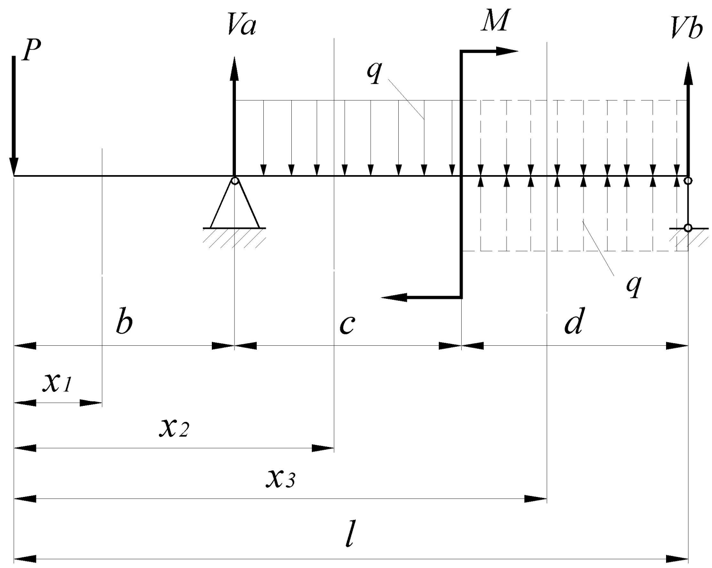
Ïðè âûâîäå îáîáùåííîãî óðàâíåíèÿ èçîãíóòîé îñè áàëêè áóäåì ñëåäîâàòü ñëåäóþùèì òðåì îñíîâíûì ïðàâèëàì:

1. Íà÷àëî êîîðäèíàò äëÿ îòñ÷åòà ïåðåìåùåíèé ìîæíî ïðèíèìàòü òîëüêî íà îäíîì èç êîíöîâ áàëêè;

2. Ëþáóþ ðàñïðåäåëåííóþ íàãðóçêó ñëåäóåò ïðîäîëæèòü äî êîíöà áàëêè, ïðîòèâîïîëîæíîìó íà÷àëó êîîðäèíàò;

3. Èíòåãðèðîâàíèå äèôôåðåíöèàëüíîãî óðàâíåíèÿ èçîãíóòîé îñè áàëêè íåîáõîäèìî âåñòè áåç ðàñêðûòèÿ ñêîáîê.

Ðàññìîòðèì áàëêó íà äâóõ îïîðàõ ñ òðåìÿ ñèëîâûìè ó÷àñòêàìè (Ðèñ 8).



Íà÷àëî êîîðäèíàò ïðèìåì íà ëåâîì êîíöå áàëêè è ïðîäîëæèì ðàñïðåäåëåííóþ íàãðóçêó äî åå ïðàâîãî êîíöà. ×òîáû íàãðóæåíèå áàëêè îñòàëîñü íåèçìåííûì "ïðèáàâëåííóþ" ðàñïðåäåëåííóþ íàãðóçêó óðàâíîâåñèì ýêâèâàëåíòíîé íàãðóçêîé, ïðèëîæåííîé íà òðåòüåì ó÷àñòêå.

Íà êàæäîì ó÷àñòêå ïðîâîäèì ïîïåðå÷íîå ñå÷åíèå. Â êàæäîì èç íèõ îïðåäåëÿåì èçãèáàþùèé ìîìåíò è çàïèñûâàåì èõ óðàâíåíèÿ â ïðàâîé ÷àñòè äèôôåðåíöèàëüíûõ óðàâíåíèé èçîãíóòîé îñè. Äèôôåðåíöèàëüíûå óðàâíåíèÿ èíòåãðèðóåì äâà ðàçà.

Ïåðâîå ñå÷åíèå. $EI(d^2y_1/dx_1^2) = –Ð(x_1 – 0)$;

$$EIQ_1 = –P(x_1 – 0)^2/2 + C_1; \qquad (a)$$

$$EIy_1 = –P(x_1 – 0)^3/6 + C_1x_1 + D_1. \qquad (b)$$

Âòîðîå ñå÷åíèå. $EI(d^2y_2/dx_2^2) = –Ð(x_2–0) + V_A(x_2–b) – q(x_2–b)^2/2$;

$$EIQ_2 = –P(x_2 – 0)^2/2 + V_A(x_2 – b)^2/2 – q(x_2 – b)^3/6 + C_2; \qquad (c)$$

$$EIy_2 = –P(x_2 – 0)^3/6 + V_A(x_2 – b)^3/6 – q(x_2 – b)^4/24 + C_2x_2 + D_2. \qquad (d)$$

Òðåòüå ñå÷åíèå. $EI(d^2y_3/dx_3^2) = –Ð(x_3–0) + V_A(x_3–b) – q(x_3–b)^2/2+$
$+ M(x_3 – c)^0 + q(x_3 – c)^2/2$;

$$EIQ_3 = –Ð(x_3–0)^2/2 + V_A(x_3–b)^2/2 –q(x_3–b)^3/6 +M(x_3–c)^1/1 +q(x_3–c)^3/6; \qquad (e)$$

$$EIy_3 = –Ð(x_3 – 0)^3/6 + V_A(x_3 – b)^3/6 – q(x_3 – b)^4/24 +$$
$$+ M(x_3 – c)^2/2 + q(x_3 – c)^4/24 + C_3x_3 + D_3. \qquad (f)$$

Âèäíî, ÷òî â ïðîöåññå èíòåãðèðîâàíèÿ íàìè ïîëó÷åíî *øåñòü* ïîñòîÿííûõ èíòåãðèðîâàíèÿ. Äîêàæåì, ÷òî íà ñàìîì äåëå íåèçâåñòíûõ ïîñòîÿííûõ èíòåãðèðîâàíèÿ â çàïèñàííûõ íàìè øåñòè óðàâíåíèÿõ (a) ... (f) òîëüêî *äâå.* Ïîäñòàâèì â óðàâíåíèÿ (à) è (ñ) $x_1 = b$ è $x_2 = b$.

$$EIQ_À \text{ (ïðè } õ_1 = b) = –P(b^2/2) + C_1; \qquad (a)$$

$$EIQ_A \text{ (ïðè } x_2 = b) = –P(b^2/2) + C_2, \qquad (c)$$

âèäíî, ÷òî $Ñ_2 = Ñ_1$.

Ïîäñòàâèì â óðàâíåíèÿ (ñ) è (å) õ$_2$ = ñ è õ$_3$ = ñ

EIQ_D (ïðè $x_2 = c$) = $–P(c^2/2) + V_A[(c \quad b)^2/2] – q[(c \quad b)^3/6] + C_2$; (c)

EIQ_D (ïðè $x_3 = c$) = $–P(c^2/2) + V_A[(c \quad b)^2/2] – q[(c \quad b)^3/6] + C_3$, (e)

âèäíî, ÷òî Ñ$_2$ = Ñ$_3$.

Òàê êàê C_1 = Ñ$_2$, à Ñ$_2$ = Ñ$_3$, òî C1 = Ñ$_2$ = Ñ$_3$ = Ñ. Àíàëîãè÷íûå ïîäñòàíîâêè ñäåëàåì è â óðàâíåíèÿ ïðîãèáîâ.

Ïîäñòàâèì â óðàâíåíèÿ (b) è (d) x_1= b è õ$_3$ = b:

Ely$_A$ (ïðè $x_1 = b$) = 0 = –Ð($b^3/6$) + Ñb + D_1;

Ely$_A$ (ïðè $x_2 = b$) = 0 = –Ð($b^3/3$) + Ñb + D_2, âèäíî, ÷òî $D_1 = D_2$.

Ïîäñòàâèì â óðàâíåíèÿ (d) è (f) õ$_2$ = ñ è õ$_3$ = c:

Ely$_D$ (ïðè õ$_2$ = ñ) = –Ð(ñ^3/6) + $V_A(c – d)^3/6 – q(c – d)^4/24$ + Ññ + D_2:

Ely$_D$ (ïðè õ$_3$ = ñ) = –Ð(ñ^3/6) + V$_À$(ñ – b)^3/6 – $q(c – b)^4/24$ + Ññ + D_3,

âèäíî, ÷òî $D_2 = D_3$.

Òàê êàê $D_1 = D_2$, à $D_2 = D_3$, òî $D_1 = D_2 = D_3 = D$. Àíàëèç óðàâíåíèé (a)...(f) ïîçâîëÿåò ñäåëàòü ñëåäóþùèå îáîáùàþùèå âûâîäû:

1. Â óðàâíåíèÿõ, ïîëó÷åííûõ äëÿ îïðåäåëåíèÿ óãëîâ ïîâîðîòîâ:

– *èìååòñÿ ïîñòîÿííàÿ èíòåãðèðîâàíèÿ Ñ = EIQ_0 [ñì. ôîðìóëó (7.5)];*

– *ìîìåíòó Ì ñîîòâåòñòâóåò ìíîæèòåëü âèäà (õ–à)^1 /1!;*

– *ñîñðåäîòî÷åííîé ñèëå Ð ñîîòâåòñòâóåò ìíîæèòåëü âèäà (õ–à)^2/2!;*

□ *ðàñïðåäåëåííîé íàãðóçêå ñîîòâåòñòâóåò ìíîæèòåëü âèäà (õ –*
– à)^3/3!;

2. Â óðàâíåíèÿõ, ïîëó÷åííûõ äëÿ îïðåäåëåíèÿ ïðîãèáîâ:

– *èìåþòñÿ ïîñòîÿííûå èíòåãðèðîâàíèÿ Ñõ = EIQ_0x è D = Ely$_0$ [ñì. ôîðìóëó (7.5)];*

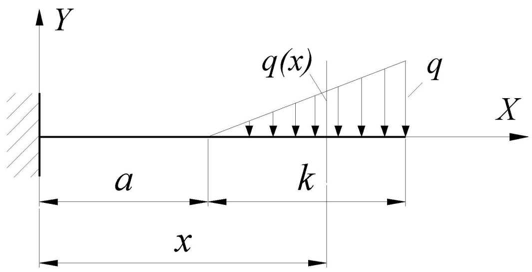
– *ìîìåíòó Ì ñîîòâåòñòâóåò ìíîæèòåëü âèäà (õ – à)^2/2!;*

– *ñîñðåäîòî÷åííîé ñèëå Ð ñîîòâåòñòâóåò ìíîæèòåëü âèäà (õ–à)^3/3!;*

□ *ðàñïðåäåëåííîé íàãðóçêå ñîîòâåòñòâóåò ìíîæèòåëü âèäà (õ – – à)^{4}/4!.*

Ïðèìå÷àíèå: *Q_0 è $ó_0$ – ñîîòâåòñòâåííî óãîë ïîâîðîòà è ïðîãèá â íà÷àëå êîîðäèíàò; à– ðàññòîÿíèå îò òî÷êè ïðèëîæåíèÿ ñèëû äî íà÷àëà êîîðäèíàò; 1! = 1; 2! = 2·1; 3! = 1·2·3; 4!= 1·2·3·4.*

Îïðåäåëèì îòäåëüíî ìíîæèòåëè äëÿ òðåóãîëüíîé ðàñïðåäåëåííîé íàãðóçêè (Ðèñ 9).



Ðèñ 9

Èçãèáàþùèé ìîìåíò â óêàçàííîì ñå÷åíèè îò òðåóãîëüíîé íàãðóçêè q ðàâåí: Ì = 1/2 q(x)(x – à)(1/3)(õ – à).

Èç óñëîâèÿ ïîäîáèÿ òðåóãîëüíèêîâ îïðåäåëèì q(x):
q(x)/q = (x – a)/k, òîãäà q(x) = [q(x – a)]/k; ñëåäîâàòåëüíî,

$$M = [q(x - a)^3]/6k.$$

Ïðè èíòåãðèðîâàíèè ïðàâîé ÷àñòè ýòîãî óðàâíåíèÿ äâà ðàçà ïîëó÷èì â óðàâíåíèè äëÿ îïðåäåëåíèÿ óãëîâ ïîâîðîòà:

$$(q/k)(x - a)^4/4!,$$

à â óðàâíåíèè äëÿ îïðåäåëåíèÿ ïðîãèáîâ:

$$(q/k)(x - a)^5/5!,$$

ãäå k– äëèíà òðåóãîëüíîé ðàñïðåäåëåííîé íàãðóçêè; 5!=1·2·3·4·5.

Íà îñíîâàíèè ïðèâåäåííîãî âûøå àíàëèçà çàïèøåì îáîáùåííûå óðàâíåíèÿ èçîãíóòîé îñè áàëêè.

Äëÿ îïðåäåëåíèÿ ïðîãèáîâ:

EIy = EIy$_0$ + EIQ$_0$x + SÌ(õ – à)^{2}/2! + SÐ(õ – à)^{3}/3! + Sq(õ – à)^{4}/4! + + (Sq/k)(õ – à)^{5}/5! (6)

Äëÿ îïðåäåëåíèÿ óãëîâ ïîâîðîòà:

$$EIQ = EIQ_0 + S\text{Ì}(\text{õ} - \text{à})^1/1! + S\text{Ð}(\text{õ} - \text{à})^2/2! + Sq(\text{õ} - \text{à})^3/3! + $$
$$+ (Sq/k)(\text{õ} - \text{à})^4/4! \qquad (7)$$

Ìåòîä íà÷àëüíûõ ïàðàìåòðîâ äàåò óðàâíåíèÿ ïðîãèáîâ è óãëîâ ïîâîðîòà, ïðè ïîìîùè êîòîðûõ ìîæíî âû÷èñëèòü ïðîãèá è óãîë ïîâîðîòà *â ëþáîì ñå÷åíèè áàëêè, à òàêæå ïîñòðîèòü åå èçîãíóòóþ îñü.* Ïðè ðåøåíèè ðÿäà çàäà÷ (íàïðèìåð, *ïðè ðàñ÷åòå ìíîãîñòóïåí÷àòûõ âàëîâ)* äîñòàòî÷íî óìåòü íàéòè ïðîãèá è óãîë ïîâîðîòà ëèøü äëÿ íåêîòîðûõ îïðåäåëåííûõ ñå÷åíèé. Â ýòîì ñëó÷àå ïðèìåíÿåòñÿ *ãðàôîàíàëèòè÷åñêèé ìåòîä.*

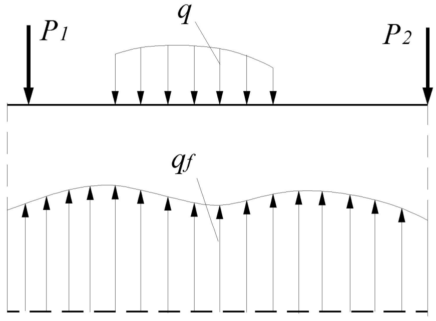
5. Ãðàôîàíàëèòè÷åñêèé ìåòîä âû÷èñëåíèÿ ïåðåìåùåíèé ïðè èçãèáå

Ìåòîä îñíîâàí íà ñõîäñòâå äèôôåðåíöèàëüíûõ çàâèñèìîñòåé, ñâÿçûâàþùèõ ïðîãèá, èçãèáàþùèé ìîìåíò è èíòåíñèâíîñòü ñïëîøíîé íàãðóçêè.

Âîçüìåì áàëêó, ê êîòîðîé ïðèëîæåíû âíåøíèå ñèëû. Çàïèøåì äèôôåðåíöèàëüíîå óðàâíåíèå åå èçîãíóòîé îñè [ôîðìóëà (2)].

$$EI(d^2y/dx^2) = \text{Ì}.$$

Ïîä ïåðâîíà÷àëüíî ïðèíÿòîé áàëêîé èçîáðàçèì åùå îäíó áàëêó òàêîé æå äëèíû, íàãðóæåííóþ ðàñïðåäåëåííîé íàãðóçêîé q_f (Ðèñ. 10).



Ðèñ 10

Îïîðàìè îáåèõ áàëîê íå çàäàåìñÿ, íî ñ÷èòàåì, ÷òî ðåàêöèè, âîçíèêàþùèå â íèõ, óðàâíîâåøèâàþò âîçäåéñòâèå ïðèëîæåííûõ ñèë. Âòîðóþ áàëêó íàçîâåì ôèêòèâíîé è âñå íàãðóçêè, ïðèëîæåííûå ê íåé, áóäåì îáîçíà÷àòü èíäåêñîì "f". Äëÿ ýòîé áàëêè âû÷èñëèì âåëè÷èíó èçãèáàþùåãî ìîìåíòà Ì$_{\text{Ñ}}$, ïîëüçóÿñü ôîðìóëîé òåîðåìû Æóðàâñêîãî.

$$d^2M_f / dx^2 = q_f.$$

Ïðîâåäåì ñîïîñòàâëåíèå çàïèñàííûõ óðàâíåíèé, ïðèíÿâ q_f = Ì. Òî åñòü, çàãðóæàåì ôèêòèâíóþ áàëêó ýïþðîé èçãèáàþùèõ ìîìåíòîâ, ïîñòðîåííîé äëÿ äåéñòâèòåëüíîé áàëêè. Òîãäà

$$EI(d^2y / dx^2) = d^2M_f / dx^2.$$

Èíòåãðèðóåì îáå ÷àñòè óðàâíåíèÿ, äîáèâàÿñü ðàâåíñòâà ïîñòîÿííûõ èíòåãðèðîâàíèÿ â ëåâîé è ïðàâîé ÷àñòÿõ.

$$EI(dy / dx) = dM_f / dx.$$

Òàê êàê $dy/dx = Q$ [ñì. ôîðìóëó (7.1)], à $dM_f/dx = Q_f$ [ñì.ôîðìóëó (5.2)], òî ïîëó÷èì $EIQ = Q_f$, èëè

$$Q = Q_f / EI. \qquad (8)$$

Óãîë ïîâîðîòà äåéñòâèòåëüíîé áàëêè (îò çàäàííîé íàãðóçêè) *ðàâåí ïîïåðå÷íîé ñèëå â òîì æå ñå÷åíèè ôèêòèâíîé áàëêè* (îò ôèêòèâíîé íàãðóçêè), *äåëåííîé íà æåñòêîñòü äåéñòâèòåëüíîé áàëêè.* Ïîñëå èíòåãðèðîâàíèÿ âòîðîé ðàç ïîëó÷èì $EIy = M_f$. Òîãäà

$$y = M_f / EI. \qquad (9)$$

Ïðîãèá äåéñòâèòåëüíîé áàëêè (îò çàäàííîé íàãðóçêè) ðàâåí èçãèáàþùåìó ìîìåíòó â òîì æå ñå÷åíèè ôèêòèâíîé áàëêè (îò ôèêòèâíîé íàãðóçêè), äåëåííîé íà æåñòêîñòü äåéñòâèòåëüíîé áàëêè.

Èç ôîðìóë (8) è (9) âûòåêàåò, ÷òî äîáèòüñÿ ðàâåíñòâà ïîñòîÿííûõ èíòåãðèðîâàíèÿ ìîæíî ïðè ñëåäóþùèõ óñëîâèÿõ:

1. Åñëè ïðîãèá äåéñòâèòåëüíîé áàëêè ðàâåí íóëþ, â òîì æå ñå÷åíèè ôèêòèâíîé áàëêè èçãèáàþùèé ìîìåíò äîëæåí áûòü ðàâåí íóëþ;

2. Åñëè óãîë ïîâîðîòà äåéñòâèòåëüíîé áàëêè ðàâåí íóëþ, â òîì æå ñå÷åíèè ôèêòèâíîé áàëêè ïîïåðå÷íàÿ ñèëà äîëæíà áûòü ðàâíà íóëþ;

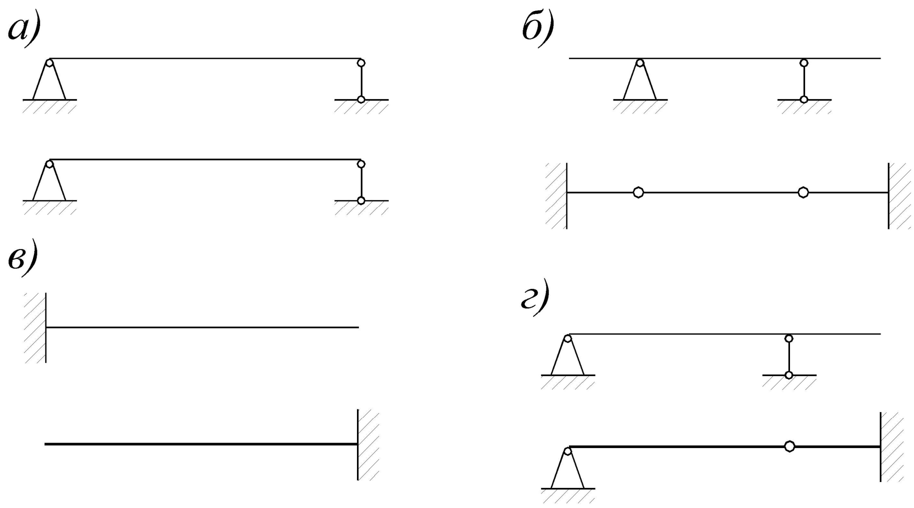
3. Åñëè ïðîãèá èëè óãîë ïîâîðîòà äåéñòâèòåëüíîé áàëêè íå ðàâíû íóëþ, òî ñîîòâåòñòâóþùèå èì Ì$_{Ñ}$ è Q_f òàêæå íå äîëæíû áûòü ðàâíû íóëþ.

Â òàáëèöå (1) ïðèâåäåíû óñëîâèÿ îáðàçîâàíèÿ ôèêòèâíûõ áàëîê.

Ò à á ë è ö à 1

Äåéñòâèòåëüíàÿ áàëêà		Ôèêòèâíàÿ áàëêà,	
Íàèìåíîâàíèå îïîðû	Óñëîâèÿ äëÿ ó è Q	Òðåáóåìûå óñëîâèÿ äëÿ Ì$_f$ è Q_f	Îòâå÷àþùèå ýòèì óñëîâèÿì îïèðàíèÿ ôèêòèâíîé áàëêè
Øàðíèðíàÿ êîíöåâàÿ îïîðà (*ïðîãèáà íåò; ïîâîðîò ñå÷åíèÿ âîçìîæåí*)	y = 0 Q ¹ 0	M_f = 0 Q_f ¹ 0	Øàðíèðíàÿ êîíöåâàÿ îïîðà (*ìîìåíòà íåò, âîçìîæíà îïîðíàÿ ðåàêöèÿ*)
Çàùåìëåííûé êîíåö áàëêè (*íåò ïðîãèáà è íåò ïîâîðîòà*)	y = 0 Q = 0	M_f = 0 Q_f = 0	Ñâîáîäíûé êîíåö áàëêè (*îòñóòñòâóþò è ñîñðåäîòî÷åííàÿ ñèëà è ìîìåíò*)
Ñâîáîäíûé êîíåö áàëêè (*âîçìîæíû è ïðîãèá è, ïîâîðîò ñå÷åíèÿ*)	y ¹ 0 Q ¹ 0	M_f ¹ 0 Q_f ¹ 0	Çàùåìëåííûé êîíåö áàëêè (*èìåþòñÿ ìîìåíò çàùåìëåíèÿ è ðåàêöèÿ*)
Ïðîìåæóòî÷íàÿ îïîðà (*ïðîãèáà íåò, ïîâîðîò ñå÷åíèÿ âîçìîæåí*)	y = 0 Q ¹ 0	M_f = 0 Q_f ¹ 0	Ïðîìåæóòî÷íûé øàðíèð (*ìîìåíòà íåò; øàðíèð ïåðåäàåò äàâëåíèå*)
Ïðîìåæóòî÷íûé øàðíèð (*âîçìîæíû è ïðîãèá è ïîâîðîò ñå÷åíèÿ*)	y ¹ 0 Q ¹ 0	M_f ¹ 0 Q_f ¹ 0	Ïðîìåæóòî÷íàÿ îïîðà (*â îïîðíîì ñå÷åíèè âîçìîæíû è ìîìåíò è ðåàêöèÿ*)

Ïîêàæåì òàêæå íàèáîëåå ðàñïðîñòðàíåííûå ñî÷åòàíèÿ äåéñòâèòåëüíûõ è ôèêòèâíûõ áàëîê. Â êàæäîé ïàðå ñîïðÿæåííûõ áàëîê ëþáàÿ ìîæåò áûòü ïðèíÿòà çà äåéñòâèòåëüíóþ, òîãäà âòîðàÿ èç íèõ áóäåò ôèêòèâíîé (Ðèñ. 11, à,á,â,ã).



Ðèñ 11

Äëÿ îïðåäåëåíèÿ óãëîâ ïîâîðîòà è ïðîãèáîâ ãðàôîàíàëèòè÷åñêèì ìåòîäîì íåîáõîäèìî ïðèäåðæèâàòüñÿ ñëåäóþùåãî ïîðÿäêà:

1. Âû÷åðòèòü ðàñ÷åòíóþ ñõåìó áàëêè;

2. Ïîñòðîèòü ýïþðó èçãèáàþùèõ ìîìåíòîâ;

3. Ïðèíÿòü îñåâóþ ëèíèþ ýïþðû Ì çà îñü ôèêòèâíîé áàëêè, è ñàìó ýïþðó çà ôèêòèâíóþ ðàñïðåäåëåííóþ íàãðóçêó q_f. Ïðè ïîëîæèòåëüíûõ çíà÷åíèÿõ èçãèáàþùåãî ìîìåíòà ñòðåëêè ðàñïðåäåëåííîé íàãðóçêè íàïðàâèòü ââåðõ, ïðè îòðèöàòåëüíûõ– âíèç;

4. Ïðèíÿòü ôèêòèâíóþ áàëêó ïî èçëîæåííûì âûøå ïðàâèëàì;

5. Îïðåäåëèòü âåëè÷èíó îïîðíûõ ðåàêöèé ôèêòèâíîé áàëêè;

6. Îïðåäåëèòü âåëè÷èíó èçãèáàþùåãî ìîìåíòà $Ì_f$ â ñå÷åíèè, â êîòîðîì òðåáóåòñÿ îïðåäåëèòü ïðîãèá;

7. Îïðåäåëèòü âåëè÷èíó ïîïåðå÷íîé ñèëû Q_f â ñå÷åíèè, óãîë ïîâîðîòà êîòîðîãî òðåáóåòñÿ îïðåäåëèòü;

8. Îïðåäåëèòü èñêîìûå óãîë ïîâîðîòà è ïðîãèá ïî ôîðìóëàì (8) è (9).

Ãðàôîàíàëèòè÷åñêèé ìåòîä îïðåäåëåíèÿ ïåðåìåùåíèé îñâîáîæäàåò îò íåîáõîäèìîñòè íàõîæäåíèÿ ïîñòîÿííûõ èíòåãðèðîâàíèÿ èëè íà÷àëüíûõ ïàðàìåòðîâ â êàæäîì ÷àñòíîì ñëó÷àå.

Ïîëó÷àþùèåñÿ â ïðîöåññå ðåøåíèÿ ôèêòèâíûå ìîìåíòû Ì$_f$ èìåþò ðàçìåðíîñòü *Íì*^{3}*, êÍì*^{3}*, ÌÍì*^{3} è ò.ä., ôèêòèâíûå ïîïåðå÷íûå ñèëû Q_f – *Íì*^{2}*, êÍì*^{2}*, ÌÍì*^{2} è ò.ä., èíòåíñèâíîñòü ôèêòèâíîé íàãðóçêè q_f – *Íì, êÍì, ÌÍì* è ò.ä.

6. Ïîòåíöèàëüíàÿ ýíåðãèÿ óïðóãîé äåôîðìàöèè ïðè èçãèáå

Êðîìå òðåõ ðàññìîòðåííûõ íàìè âûøå ñïîñîáîâ îïðåäåëåíèÿ ïåðåìåùåíèé ïðè èçãèáå ñóùåñòâóþò áîëåå îáùèå ìåòîäû, ïðèãîäíûå äëÿ îïðåäåëåíèÿ äåôîðìàöèé ëþáûõ êîíñòðóêöèé, îñíîâàííûå íà ïðèìåíåíèè *çàêîíà ñîõðàíåíèÿ ýíåðãèè.* Ïðè ñòàòè÷åñêîì íàãðóæåíèè ýëåìåíòîâ âíåøíèìè ñèëàìè äåôîðìàöèÿ äåòàëè èëè ýëåìåíòà êîíñòðóêöèè íå áóäåò ñîïðîâîæäàòüñÿ èçìåíåíèåì êèíåòè÷åñêîé ýíåðãèè ñèñòåìû, à áóäåò èìåòü ìåñòî ëèøü ïðåîáðàçîâàíèå *ïîòåíöèàëüíîé ýíåðãèè âíåøíèõ ñèë* â *ïîòåíöèàëüíóþ ýíåðãèþ óïðóãîé äåôîðìàöèè.*

Îáîçíà÷èì âåëè÷èíó ïîòåíöèàëüíîé ýíåðãèè óïðóãîé äåôîðìàöèè ÷åðåç U, a ïîòåíöèàëüíóþ ýíåðãèþ âíåøíèõ ñèë ÷åðåç U_P. Âåëè÷èíà U_P èçìåðÿåòñÿ ïîëîæèòåëüíîé ðàáîòîé ýòèõ ñèë À$_{\text{Ð}}$, à ýíåðãèÿ U áóäåò èçìåðÿòüñÿ îòðèöàòåëüíîé ðàáîòîé âíóòðåííèõ ñèë À, òàê êàê ïåðåìåùåíèÿ òî÷åê ýëåìåíòà ïðè äåôîðìàöèè ïðîèñõîäÿò â îáðàòíîì ïî îòíîøåíèþ ê âíóòðåííèì ñèëàì íàïðàâëåíèè.

Òîãäà çàêîí ñîõðàíåíèÿ ýíåðãèè ïðèìåò âèä:

$$U_P = U.$$

Çàìåíÿÿ â ýòîé ôîðìóëå âåëè÷èíû U_P è U ÷èñëåííî ðàâíûìè èì çíà÷åíèÿìè ðàáîò $À_Ð$ è –À, ïîëó÷èì èíóþ ôîðìóëèðîâêó ýòîãî çàêîíà

$$À_Ð = –À, \text{ èëè } À_Ð + À = 0.$$

Èç ýòîãî ñëåäóåò, ÷òî ïîòåíöèàëüíàÿ ýíåðãèÿ óïðóãîé äåôîðìàöèè U ÷èñëåííî ðàâíà ðàáîòå âíåøíèõ ñèë Àð, ïðîäåëàííîé èìè â ïðîöåññå äåôîðìàöèè.

$$U = A_P. \qquad (10)$$

Ðàíåå áûëî ïîêàçàíî, ÷òî ïðè ðàñòÿæåíèè èëè ñæàòèè ïîòåíöèàëüíóþ ýíåðãèþ óïðóãîé äåôîðìàöèè ìîæíî ïîäñ÷èòàòü ïî ôîðìóëå U = (1/2)ÐDl. Àíàëîãè÷íûå çàâèñèìîñòè ìîæíî çàïèñàòü è äëÿ äðóãèõ âèäîâ äåôîðìàöèé.

Äëÿ ÷èñòîãî ñäâèãà: U = (1/2)TDs; äëÿ êðó÷åíèÿ: U = (1/2)$Ì_{êð}$j ; äëÿ èçãèáà: U = (1/2)ÌQ.

Èç ïðèâåäåííûõ âûøå ôîðìóë ñëåäóåò, ÷òî ïîòåíöèàëüíàÿ ýíåðãèÿ óïðóãîé äåôîðìàöèè ðàâíà ïðîèçâåäåíèþ âíóòðåííåãî óñèëèÿ íà ïåðåìåùåíèå òîãî ñå÷åíèÿ, ãäå îíî ïðèëîæåíî. Íàçîâåì òåðìèíîì *"îáîáùåííàÿ ñèëà Ð"* óñèëèå, âûçûâàþùåå äåôîðìàöèþ. Ïåðåìåùåíèå, ñîîòâåòñòâóþùåå *îáîáùåííîé ñèëå,* áóäåì íàçûâàòü *îáîáùåííûì ïåðåìåùåíèåì* d. Òîãäà *ïîòåíöèàëüíàÿ ýíåðãèÿ óïðóãîé äåôîðìàöèè ðàâíà ïîëîâèíå ïðîèçâåäåíèÿ îáîáùåííîé ñèëû íà îáîáùåííîå ïåðåìåùåíèå:*

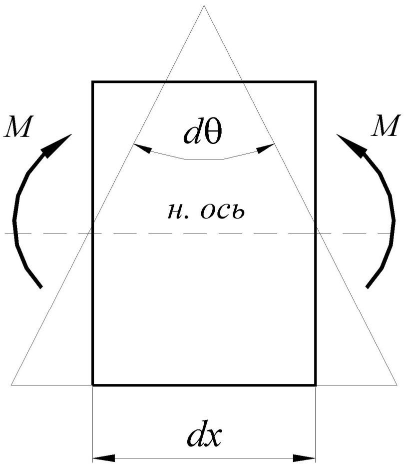
$$U = (1/2)Ðd. \qquad (11)$$

Âû÷èñëåíèå ïîòåíöèàëüíîé ýíåðãèè óïðóãîé äåôîðìàöèè ïðè äåéñòâèè íåñêîëüêèõ ñèë ñëåäóåò ïðîèçâîäèòü ïî ôîðìóëå Êëàïåéðîíà:

$$U = (1/2)Ð_1d_1 + (1/2)Ð_2d_2 + (1/2)Ð_3d_3 + ... + (1/2)P_nd_n. \qquad (12)$$

Òàêèì îáðàçîì, *ïîòåíöèàëüíàÿ ýíåðãèÿ óïðóãîé äåôîðìàöèè, âîçíèêàþùàÿ â ðåçóëüòàòå äåéñòâèÿ íåñêîëüêèõ îáîáùåííûõ ñèë,* *ðàâíà ïîëîâèíå ñóììû ïðîèçâåäåíèé îáîáùåííûõ ñèë íà îáîáùåííûå ïåðåìåùåíèÿ, îáðàçóþùèåñÿ îò ñîâìåñòíîãî äåéñòâèÿ îáîáùåííûõ ñèë.*

Âîçüìåì ó÷àñòîê áàëêè äëèíîé dx ñ äåéñòâóþùèìè íà íåãî èçãèáàþùèìè ìîìåíòàìè (Ðèñ 12).



Ðèñ 12.

Ïîä äåéñòâèåì èçãèáàþùèõ ìîìåíòîâ ñå÷åíèÿ, îãðàíè÷èâàþùèå âûäåëåííûé ó÷àñòîê, ïîâîðà÷èâàþòñÿ è îáðàçóþò ìåæäó ñîáîé óãîë dQ. Ïîëüçóÿñü ôîðìóëîé (11), ïîëó÷èì:

dU = (1/2)MdQ; ïîäñòàâèì âìåñòî Q åãî çíà÷åíèå èç ôîðìóëû (3).

dU = (1/2)Md[(1/EI) òÌdx + Ñ] = (1/2EI)Ì^{2}dx, èëè dU = M^{2}dx /(2EI).

$$U = ò_0^1(Ì^2\ /2EI)\ dx. \qquad (13)$$

Èíòåãðèðóÿ ïî âñåé äëèíå áàëêè, ïîëó÷èëè ôîðìóëó (13) äëÿ âû÷èñëåíèÿ ïîòåíöèàëüíîé ýíåðãèè óïðóãîé äåôîðìàöèè ïðè èçãèáå.

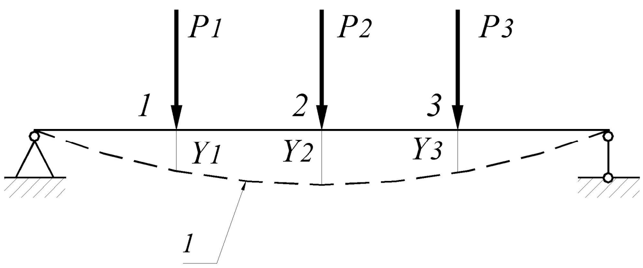
7. Òåîðåìà Êàñòèëüÿíî

Ýòà òåîðåìà ÿâëÿåòñÿ îñíîâîé äëÿ ïîëó÷åíèÿ ýíåðãåòè÷åñêèõ ìåòîäîâ îïðåäåëåíèÿ ïåðåìåùåíèé ïðè èçãèáå. Îíà äîêàçûâàåò, ÷òî

– *÷àñòíàÿ ïðîèçâîäíàÿ îò ïîòåíöèàëüíîé ýíåðãèè óïðóãîé* *äåôîðìàöèè ïî ñèëå, ïðèëîæåííîé â êàêîì–ëèáî ñå÷åíèè áàëêè– åñòü ïðîãèá ýòîãî ñå÷åíèÿ;*

– ÷àñòíàÿ ïðîèçâîäíàÿ îò ïîòåíöèàëüíîé ýíåðãèè óïðóãîé äåôîðìàöèè ïî ìîìåíòó, ïðèëîæåííîìó â êàêîì–ëèáî ñå÷åíèè áàëêè– åñòü óãîë ïîâîðîòà ýòîãî ñå÷åíèÿ.

Â òî÷êàõ 1, 2 è 3 áàëêè ñòàòè÷åñêè ïðèëîæèì ñèëû P_1, P_2 è $Ð_3$. (Ðèñ. 13).



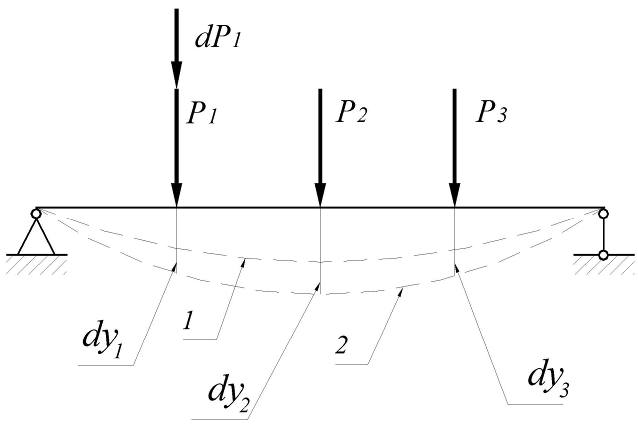
Ðèñ 13

Ïîä äåéñòâèåì ýòèõ ñèë áàëêà èçîãíåòñÿ è çàéìåò ïîëîæåíèå 1.

Ïðè ýòîì áóäåò ñîâåðøåíà ðàáîòà

$$À_{Ð} = (1/2)P_1y_1 + (1/2)Ð_2ó_2 + (1/2)Ð_3ó_3.$$

Ïåðåâåäåì áàëêó, íå íàðóøàÿ ðàâíîâåñèÿ â ïîëîæåíèå 2. Äîáàâèì äëÿ ýòîãî ê ñèëå P_1 áåñêîíå÷íî ìàëóþ äîáàâêó dP_1 .(Ðèñ 14).

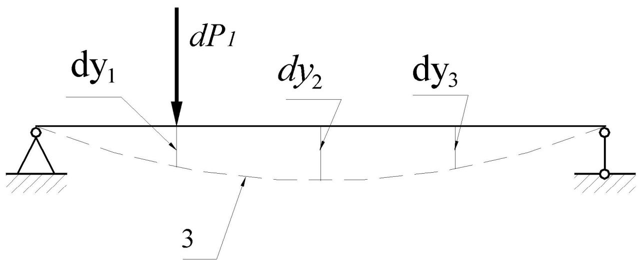


Ðèñ 14

Ïðè ïåðåõîäå áàëêè èç ïîëîæåíèÿ 1 â ïîëîæåíèå 2 âñå ñèëû Ð îïóñòÿòñÿ è âñëåäñòâèå ýòîãî èõ ïîòåíöèàëüíàÿ ýíåðãèÿ óìåíüøèòñÿ. Ïðè ýòîì *óìåíüøåíèå ýíåðãèè âíåøíèõ ñèë* dU_P ïðåîáðàçóåòñÿ â *óâåëè÷åíèå ïîòåíöèàëüíîé ýíåðãèè óïðóãîé äåôîðìàöèè áàëêè dU.* Òàê êàê ýíåðãèÿ U ÿâëÿåòñÿ ôóíêöèåé íåñêîëüêèõ íåçàâèñèìûõ ïåðåìåííûõ $U = f(P_1, Ð_2, Ð_3)$, äèôôåðåíöèàë åå íàõîäèòñÿ ñ ïîìîùüþ *ðÿäà Òýéëîðà:*

$$dU = (¶U\ /¶P_1)dP_1. \qquad (a)$$

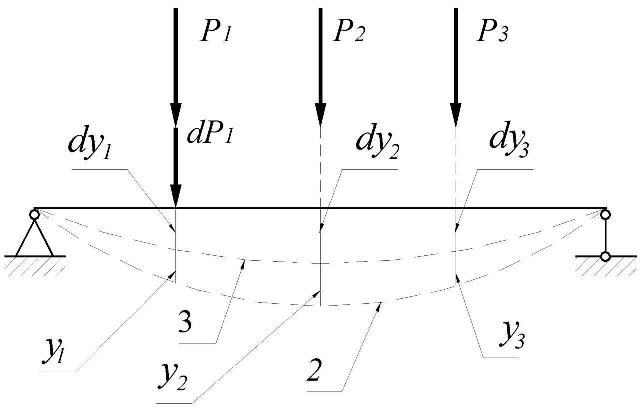
Îïðåäåëèì ýòîò äèôôåðåíöèàë äðóãèì ïóòåì, èçìåíèâ ïîðÿäîê ïðèëîæåíèÿ âíåøíèõ ñèë. Íàãðóçèì íàøó áàëêó âíà÷àëå ñèëîé dP_1.(Ðèñ 15).



Ðèñ 15

Áàëêà çàéìåò ïîëîæåíèå 3 è ïðîãèáû åå â òî÷êàõ 1, 2 è 3 áóäóò ñîîòâåòñòâåííî dy_1, dy_2 è dy_3. Ïðè ýòîì ïîëó÷èì ðàáîòó îò ñòàòè÷åñêè ïðèëîæåííîé ñèëû dP_1: $dA_P = (1/2)dP_1dy_1$.()

Ïîñëå ýòîãî íà÷íåì ïîñòåïåííî íàãðóæàòü áàëêó îäíîâðåìåííî âîçðàñòàþùèìè íàãðóçêàìè P_1, $Ð_2$ è P_3 âñëåäñòâèå ÷åãî áàëêà îïÿòü çàéìåò ïîëîæåíèå 2 (Ðèñ. 16).



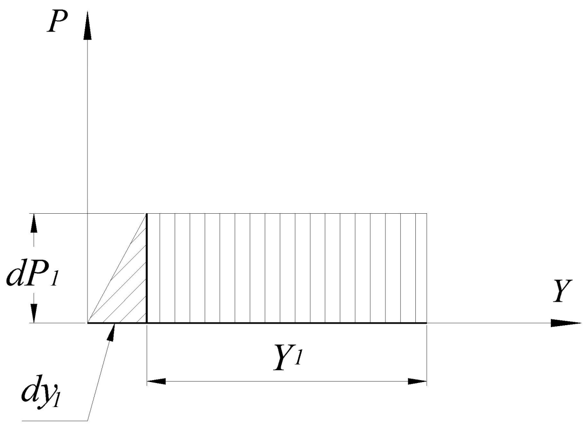
Ðèñ.16

Ïðè ýòîì áóäåò ñîâåðøåíà óæå íàéäåííàÿ ðàíåå ðàáîòà.

$$A_P = (1/2)P_1y_1 + (1/2)P_2y_2 + (1/2)P_3y_3.$$

Êðîìå òîãî, ïðîèçâåäåò ðàáîòó óæå íàõîäÿùàÿñÿ íà áàëêå ñèëà dP_1. Îíà ïðîéäåò ïóòü y_1, îñòàâàÿñü ïðè íàãðóæåíèè áàëêè ñèëàìè P_1, Ð$_2$ è P_3 ïîñòîÿííîé.

Ñëåäîâàòåëüíî, êàê ýòî âèäíî èç ðèñóíêà 17, ðàáîòà ñèëû dP_1 íà ïåðåìåùåíèè y_1 âûðàçèòñÿ ïëîùàäüþ ïðÿìîóãîëüíèêà è áóäåò ðàâíà dP_1y_1.



Ðèñ 17

Òîãäà âñÿ ðàáîòà, çàòðà÷åííàÿ íà ïåðåâîä áàëêè â ïîëîæåíèå 2, âûðàçèòñÿ ñëåäóþùåé ôîðìóëîé:

$$\text{À}_2 = dA_P+A_P+dP_1y_1 = (1/2)dP_1dy_1+(1/2)P_1y_1+(1/2)P_2y_2+(1/2)P_3y_3+dP_1y_1.$$

Òàê êàê äëÿ ïåðåâîäà áàëêè â ïîëîæåíèå 1 áûëà çàòðà÷åíà ðàáîòà $À_{Ð}$, à â ïîëîæåíèå 2– ðàáîòà $À_2$, òî

$$dA_P = dU = À_2 – À_Ð = dA_P + dP_1y_1 = (1/2)dP_1dy_1 + dP_1y_1.$$

Â ýòîì âûðàæåíèè ïðåíåáðåãàåì ñëàãàåìûì âòîðîãî ïîðÿäêà ìàëîñòè $(1/2)dP_1dy_1$:

$$dU = dP_1y_1. \quad (b)$$

Ïðèðàâíèâàåì ïðàâûå ÷àñòè ôîðìóë (à) è (b):

$(¶U/¶P_1)dP_1 = dP_1y_1$, òîãäà ìîæåì çàïèñàòü:

$$y_1 = ¶U/¶P_1. \quad (14)$$

Ýòî è òðåáîâàëîñü äîêàçàòü.

Àíàëîãè÷íûìè ðàññóæäåíèÿìè ìîæíî ïîëó÷èòü ôîðìóëó äëÿ îïðåäåëåíèÿ óãëà ïîâîðîòà ñå÷åíèÿ.

$$Q_1 = ¶U/¶M_1. \quad (15)$$

Ðàññìîòðåííàÿ íàìè òåîðåìà áûëà îïóáëèêîâàíà â 1875 ãîäó.

Ïîäñòàâèì â ýòè ôîðìóëû çíà÷åíèå ïîòåíöèàëüíîé ýíåðãèè óïðóãîé äåôîðìàöèè èç ôîðìóëû (13):

$$y_1 = ¶[ò_0^1(Ì^2/2EI)dx]/¶P_1.$$

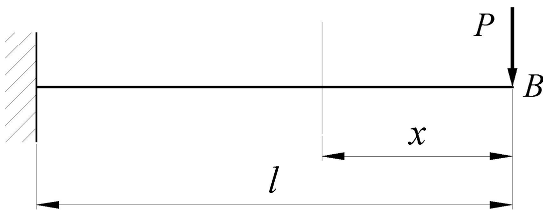
Â ýòîé ôîðìóëå ìû èìååì äåëî ñ äèôôåðåíöèðîâàíèåì îïðåäåëåííîãî èíòåãðàëà ïî ïàðàìåòðó, òàê êàê èçãèáàþùèé ìîìåíò Ì ÿâëÿåòñÿ ôóíêöèåé è P_1 è àáñöèññû ñå÷åíèÿ õ. Â ýòîì ñëó÷àå èíòåãðèðîâàíèå ïðîèçâîäèòñÿ ïî õ, à äèôôåðåíöèðîâàíèå ïî P_1. Òàê êàê ïðåäåëû èíòåãðàëà ïîñòîÿííû, äèôôåðåíöèðóåì ïîäûíòåãðàëüíóþ ôóíêöèþ:

$$y_1 = ò_0^1(Ì^2dx/EI)(¶M/¶P_1). \quad (16)$$

Òàêèì æå îáðàçîì ïîëó÷èì ôîðìóëó äëÿ îïðåäåëåíèÿ óãëîâ ïîâîðîòà:

$$Q_1 = \int_0^1 (Ì^2 dx / EI)(¶M / ¶M_1). \qquad (17)$$

Íà ïðèìåðå êîíñîëüíîé áàëêè ðàññìîòðèì îïðåäåëåíèå ïðîãèáà â ñå÷åíèè, ïðîõîäÿùèì ÷åðåç òî÷êó Â (Ðèñ. 18).



Ðèñ. 18

Â ïðîâåäåííîì ñå÷åíèè Ì = –Ðõ. Ïîäñòàâèì åãî â ôîðìóëó (16):

$$y_B = \int_0^1 [(–Ðõ)^2 dx / EI][¶(–Px) / ¶P], \text{ èëè}$$

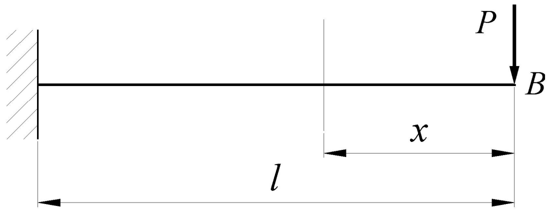
$$y_B = \int_0^1 [(–Ðõ)^2 dx / EI](–x) = (P / EI)(x^3 / 3)\Big|_0^1 = Pl^3 / 3EI.$$

Â äàííîì ïðèìåðå ïðîãèá â ñå÷åíèè îïðåäåëåí â ì.

8. Òåîðåìà î âçàèìíîñòè ðàáîò

Ïî ýòîé òåîðåìå *ðàáîòà ñèëû P_1 íà ïåðåìåùåíèÿõ, âûçâàííûõ ñèëîé $Ð_2$, ðàâíà ðàáîòå ñèëû P_2 íà ïåðåìåùåíèÿõ âûçâàííûõ ðàáîòîé ñèëû P_1.*

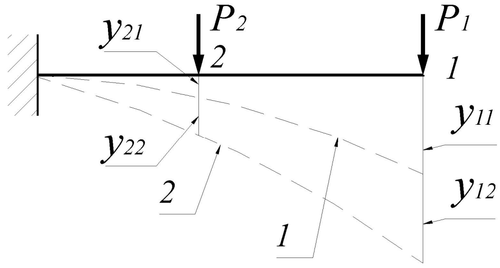
Ïðèëîæèì ê êîíñîëüíîé áàëêå â òî÷êå 1 ñòàòè÷åñêè ñèëó P_1 (Ðèñ. 19).



Ðèñ 19

Áàëêà çàéìåò ïîëîæåíèå 1. Ïðè ýòîì â òî÷êå 1 ïîëó÷èì ïðîãèá y_{11} (ïðîãèá òî÷êè 1 îò äåéñòâèÿ ñèëû P_1), à â òî÷êå 2– ïðîãèá y_{21} (ïðîãèá òî÷êè 2 îò äåéñòâèÿ ñèëû P_1).

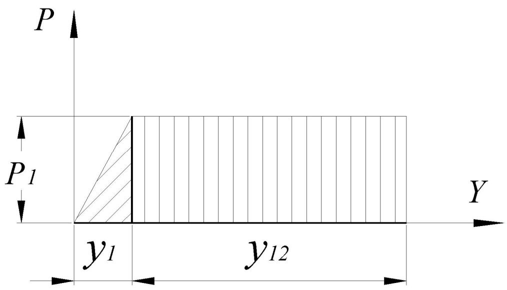
Ïðè ýòîì ñèëà P_1 íà ïåðåìåùåíèè ó$_{11}$ ñîâåðøèò ðàáîòó, ðàâíóþ (1/2)P_1ó$_{11}$. Ïîñëå ïðèëîæåíèÿ â òî÷êå 2 ñèëû P_2 áàëêà ïåðåìåñòèòñÿ â ïîëîæåíèå 2 (Ðèñ. 20).



(Ðèñ. 20).

Ïðè ýòîì ïîëó÷èì ïåðåìåùåíèÿ òî÷êè 2 îò äåéñòâèÿ ñèëû Ð$_1$(ó$_{22}$) è òî÷êè 1 îò äåéñòâèÿ ñèëû Ð$_2$(ó$_{12}$).

Â ïðîöåññå ïåðåâîäà áàëêè â ïîëîæåíèå 2 ñèëà Ð$_2$ ñîâåðøèò ðàáîòó íà ïåðåìåùåíèè ó$_{22}$, ðàâíóþ (1/2)P_2y_{22} è ñèëà P_1, îñòàâàÿñü ïîñòîÿííîé, ñîâåðøèò ðàáîòó íà ïåðåìåùåíèè– P_1ó$_{12}$ (ýòà ðàáîòà ðàâíà ïëîùàäè ïðÿìîóãîëüíèêà) (Ðèñ. 21).

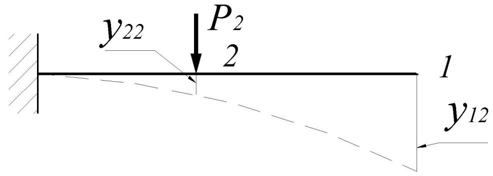


Ðèñ. 21

Ñóììèðóÿ ðàáîòó ñèë, ïîëó÷èì âåëè÷èíó ðàáîòû, íåîáõîäèìóþ äëÿ ïåðåâîäà áàëêè â ïîëîæåíèå 2:

$$A_2 = (1/2)P_1y_{11} + (1/2)P_2y_{22} + P_1y_{12}. \qquad \text{(a)}$$

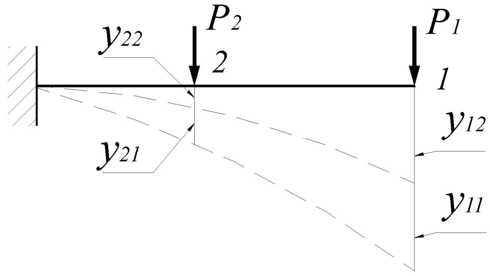
Èçìåíèì ïîðÿäîê ïðèëîæåíèÿ âíåøíèõ ñèë. Ïðèëîæèì âíà÷àëå ñèëó Ð$_2$ â òî÷êå 2. Áàëêà çàéìåò ïîëîæåíèå 3 (Ðèñ. 22).



Ðèñ. 22

Ïîëó÷èì ïðîãèáû ó$_{22}$ è y_{12}. Ñèëà Ð$_2$ ñîâåðøèò ðàáîòó íà ïåðåìåùåíèè ó$_{22}$ ðàâíóþ (1/2)Ð$_2$ó$_{22}$.

Ïîñëå ýòîãî ïðèëîæèì â òî÷êå 1 ñèëó P_1 (Ðèñ. 23).



Ðèñ. 23

Ïîëó÷èì ïðîãèáû ó$_{11}$ è ó$_{21}$. Ñèëà P_1 ñîâåðøèò ðàáîòó íà ïåðåìåùåíèè $ó_{22}$ ðàâíóþ $P_1ó_{11}$. Ñèëà P_1, îñòàâàÿñü ïîñòîÿííîé, ñîâåðøèò ðàáîòó, íà ïåðåìåùåíèè y_{21}, ðàâíóþ $Ð_2ó_{21}$. Â èòîãå áàëêà, êàê è â ïðîøëîì ñëó÷àå, îêàæåòñÿ â ïîëîæåíèè 2.

Ñóììàðíàÿ ðàáîòà ïðè íîâîì ïîðÿäêå ïðèëîæåíèÿ ñèë $À_2$ îïðåäåëÿåòñÿ èç ñëåäóþùåãî âûðàæåíèÿ:

$$A_2 = (1/2)P_1y_{11} + (1/2)P_2y_{22} + P_2y_{21}. \quad (b)$$

Ïðèðàâíèâàÿ ïðàâûå ÷àñòè ôîðìóë (à) è (b), ïîëó÷èì:

$$P_1y_{12} = P_2y_{21}. \quad (18)$$

Ýòî è òðåáîâàëîñü äîêàçàòü.

Òåîðåìó î âçàèìíîñòè ðàáîò ìîæíî ñôîðìóëèðîâàòü èíà÷å: *ðàáîòà ïåðâîé ñèëû P_1 ïðè äåéñòâèè âòîðîé P_2 ðàâíà ðàáîòå âòîðîé ñèëû ïðè äåéñòâèè ïåðâîé*. Â ñëó÷àå, êîãäà $P_1 = P_2$, ïîëó÷èì òåîðåìó *î âçàèìíîñòè ïåðåìåùåíèé*:

$$y_{12} = y_{21}. \quad (19)$$

Èç ôîðìóëû (19) âûòåêàåò, ÷òî *ïðîãèá â òî÷êå 1, âûçâàííûé ñèëîé, ïðèëîæåííîé â òî÷êå 2, ðàâåí ïðîãèáó â òî÷êå 2, âûçâàííîìó òàêîé æå ñèëîé, ïðèëîæåííîé â òî÷êå 1.*

Òåîðåìà î âçàèìíîñòè ðàáîò øèðîêî èñïîëüçóåòñÿ â ïðàêòèêå ëàáîðàòîðíûõ èññëåäîâàíèé.

9. Òåîðåìà Ìàêñâåëëà – Ìîðà

Ïî ýòîé òåîðåìå âû÷èñëåíèå ÷àñòíûõ ïðîèçâîäíûõ èçãèáàþùèõ ìîìåíòîâ â ôîðìóëàõ (16) è (17) ìîæíî çàìåíèòü âû÷èñëåíèåì èçãèáàþùèõ ìîìåíòîâ îò åäèíè÷íûõ ñèë èëè ìîìåíòà.

Ñ ïîìîùüþ òåîðåìû Êàñòèëüÿíî áûëè ïîëó÷åíû ôîðìóëû (16) è (17) äëÿ îïðåäåëåíèÿ ïðîãèáîâ è óãëîâ ïîâîðîòîâ ñå÷åíèé áàëîê. Â íèõ âõîäÿò ÷àñòíûå ïðîèçâîäíûå ¶M/¶P_1 è ¶M/¶M_1. Âûÿñíèì, ÷òî îíè ñîáîé ïðåäñòàâëÿþò.

Ëþáîå óðàâíåíèå èçãèáàþùèõ ìîìåíòîâ ïðåäñòàâëÿåò ñîáîé ëèíåéíóþ ôóíêöèþ îò âíåøíèõ íàãðóçîê è ìîæåò áûòü â îáùåì âèäå записано следующим образом:

$$\text{Ì} = a_1P_1 + a_2P_2 + \dots + b_1M_1 + b_2M_2 + \dots + c_1q_1 + c_2q_2 + \dots$$

Â ýòîì óðàâíåíèè êîýôôèöèåíòû a_1, a_2, b_1, b_2, c_1 è ò.ä. ÿâëÿþòñÿ ôóíêöèÿìè ïðîëåòà áàëêè, ðàññòîÿíèé òî÷åê ïðèëîæåíèÿ ñèë è ìîìåíòîâ îò îïîð è àáñöèññû ñå÷åíèÿ. Äîïóñòèì, ÷òî íàì íàäî îïðåäåëèòü ïðîãèá òî÷êè ïðèëîæåíèÿ ñèëû P_1. Òîãäà ÷àñòíàÿ ïðîèçâîäíàÿ ¶Ì/¶P_1 = a_1, òàê êàê âñå îñòàëüíûå ÷ëåíû óðàâíåíèÿ èçãèáàþùèõ ìîìåíòîâ ÿâëÿþòñÿ ïðè ýòîì äèôôåðåíöèðîâàíèè ïîñòîÿííûìè âåëè÷èíàìè. Íî a_1 ìîæíî ðàññìàòðèâàòü êàê ÷èñëåííóþ âåëè÷èíó ìîìåíòà Ì â ëþáîì ñå÷åíèè áàëêè îò äåéñòâèÿ ñèëû Ð = 1. Âåäü åñëè ïîñòàâèòü â çàïèñàííîå íàìè óðàâíåíèå ìîìåíòîâ P_1 = 1 è ïðèðàâíÿòü âñå îñòàëüíûå íàãðóçêè íóëþ, ïîëó÷èì, ÷òî Ì = a_1. Àíàëîãè÷íî ïðè M_1 = 1, ïîëó÷èì Ì = b_1 è ò.ä.

Ñëåäîâàòåëüíî, äëÿ âû÷èñëåíèÿ ïåðåìåùåíèÿ d (îíî ìîæåò áûòü ïðîãèáîì èëè óãëîì ïîâîðîòà) ëþáîãî ñå÷åíèÿ áàëêè íåîáõîäèìî:

1. Çàïèñàòü óðàâíåíèå èçãèáàþùåãî ìîìåíòà â ýòîì ñå÷åíèè îò çàäàííûõ âíåøíèõ ñèë;

2. Íàéòè âûðàæåíèå äëÿ èçãèáàþùåãî ìîìåíòà îò ñîîòâåòñòâóþùåé åäèíè÷íîé íàãðóçêè, ïðèëîæåííîé â ñå÷åíèè, â êîòîðîì òðåáóåòñÿ íàéòè ïåðåìåùåíèå d.

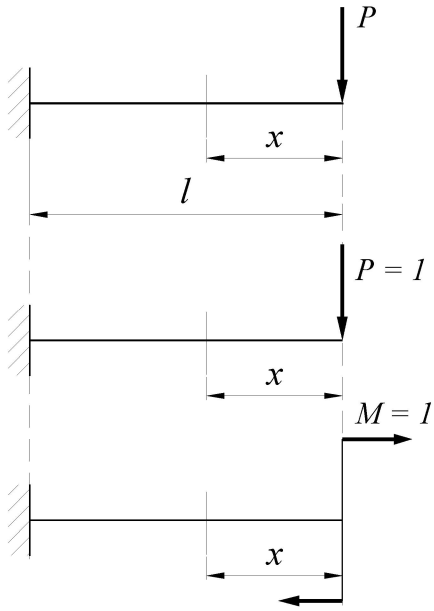
Òîãäà ýòî ïåðåìåùåíèå ìîæíî íàéòè ïî ôîðìóëå:

$$d = \int_{0}^{1} \text{Ì}\,\text{Ì}^{0}\ /EI)dx. \qquad (20)$$

Ýòà ôîðìóëà áûëà ïðåäëîæåíà Ìàêñâåëëîì â 1864 ãîäó è ââåäåíà â ïðàêòèêó ðàñ÷åòîâ Ìîðîì â 1874 ãîäó.

Åñëè â ôîðìóëå (20) ïîä d ïîäðàçóìåâàåòñÿ ïðîãèá, òî ìîìåíò $Ì^0$ íàäî âû÷èñëÿòü îò ñèëû Ð = 1, ïðèëîæåííîé â òî÷êå, äëÿ êîòîðîé íàäî íàéòè ïðîãèá. Ïðè âû÷èñëåíèè óãëà ïîâîðîòà ñå÷åíèÿ â êà÷åñòâå åäèíè÷íîé íàãðóçêè â ýòîì ñå÷åíèè íàäî ïðèëîæèòü ìîìåíò Ì = 1.

Íà ïðèìåðå êîíñîëüíîé áàëêè (Ðèñ. 24) îïðåäåëèì ïðîãèá è óãîë ïîâîðîòà êîíöåâîãî ñå÷åíèÿ.



Ðèñ. 24

Çàïèñûâàåì óðàâíåíèå èçãèáàþùåãî ìîìåíòà îò ñèëû Ð:

$$Ì = –Ðõ.$$

Çàïèñûâàåì óðàâíåíèå èçãèáàþùåãî ìîìåíòà îò ñèëû Ð = 1:

$$Ì^0 = –õ.$$

Ïîäñòàâëÿåì Ì è $Ì^0$ â ôîðìóëó (20):

$$y = \int_0^1 [(–Ðõ)(–x)/EI]dx = (P/EI)(x^3/3)\Big|_0^1 = Pl^3/3EI.$$

4. Çàïèñûâàåì óðàâíåíèå èçãèáàþùåãî ìîìåíòà îò ïàðû ñèë Ì=1: Ì = –1.

5. Ïîäñòàâëÿåì Ì è Ì^0 â ôîðìóëó (7.20):

$$Q = \int_0^1 [(-Ðõ)(-1)/EI]\,dx = (P/EI)(x^2/2)\Big|_0^1 = Pl^2/2EI.$$

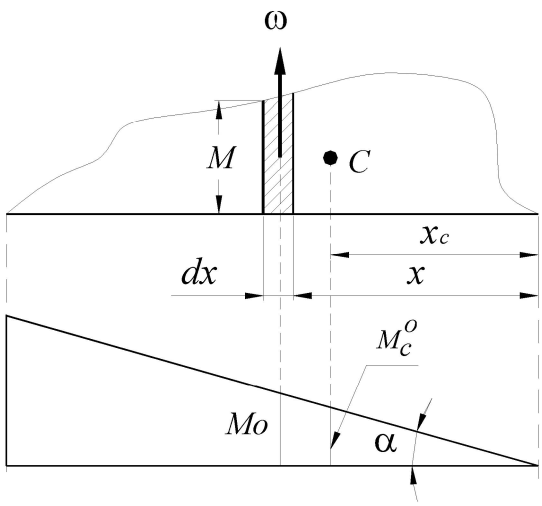
Â äàííîì ïðèìåðå ïðîãèá â ñå÷åíèè îïðåäåëåí â ì, à óãîë ïîâîðîòà â ðàä.

10. Ìåòîä Âåðåùàãèíà

Òàê êàê åäèíè÷íîé íàãðóçêîé áûâàåò ëèáî ñîñðåäîòî÷åííàÿ ñèëà, ëèáî ïàðà ñèë, òî ýïþðà èçãèáàþùèõ ìîìåíòîâ Ì^0 áóäåò âñåãäà îãðàíè÷èâàòüñÿ ïðÿìûìè ëèíèÿìè.

Àíäðåé Íèêîëàåâè÷ Âåðåùàãèí, áóäó÷è ñòóäåíòîì Ìîñêîâñêîãî èíñòèòóòà èíæåíåðîâ æåëåçíîäîðîæíîãî òðàíñïîðòà, ïðåäëîæèë ãðàôîàíàëèòè÷åñêèé ìåòîä âû÷èñëåíèÿ èíòåãðàëà Ìîðà.

Ïóñòü ýïþðà Ì îò âíåøíèõ ñèë èìååò êðèâîëèíåéíîå î÷åðòàíèå, à ýïþðà Ì^0– ïðÿìîëèíåéíîå.



Ðèñ 25

Êàê âèäíî èç ðèñóíêà (Ðèñ. 25),ïëîùàäü ýëåìåíòà äèàãðàììû dx è âûñîòîé, ðàâíîé èçãèáàþùåìó ìîìåíòó â äàííîì ñå÷åíèè, dw = Mdx. Âèäíî òàêæå, ÷òî îðäèíàòà íà ýïþðå îò åäèíè÷íîé ñèëû Ì^0 = xtga.

Òîãäà ôîðìóëà èíòåãðàëà Ìîðà (20) ïðèìåò âèä:

$$d = (tga / EI)Ò_0^1 xdw, \text{ èëè } d = (tga / EI)S_Y.$$

Ñ äðóãîé ñòîðîíû ñòàòè÷åñêèé ìîìåíò ñå÷åíèÿ $S_Y = õ_Ñw$,

ãäå $õ_Ñ$– êîîðäèíàòà öåíòðà òÿæåñòè ýïþðû Ì;

w– ïëîùàäü ýïþðû Ì.

 ðåçóëüòàòå ïîäñòàíîâêè çíà÷åíèÿ S_Y èíòåãðàë Ìîðà ïðèìåò âèä: $d = (x_C tga \cdot w) / EI$, íî $x_C tga = M^0_Ñ$, òîãäà îêîí÷àòåëüíî ïîëó÷èì:

$$d = Ì^0_Ñ (w / EI), \tag{21}$$

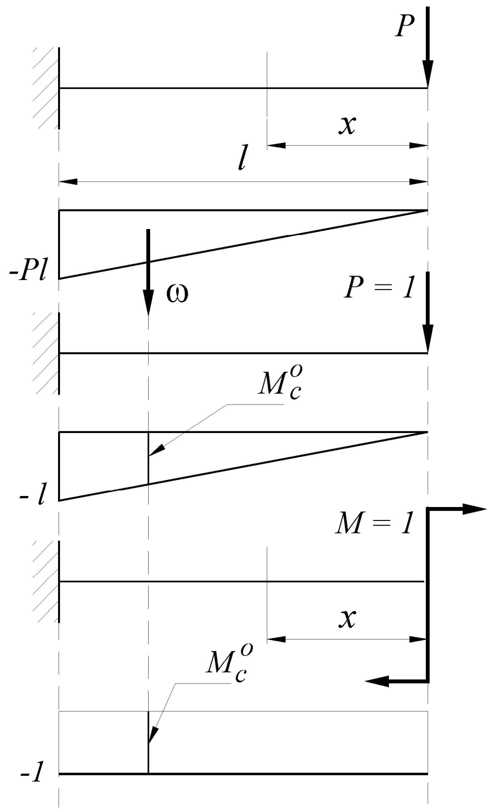
ãäå w– ïëîùàäü ýïþðû èçãèáàþùèõ ìîìåíòîâ, ïîñòðîåííîé îò äåéñòâèÿ âíåøíèõ ñèë, ïðèëîæåííûõ ê áàëêå;

$Ì^0_Ñ$– îðäèíàòà ýïþðû èçãèáàþùèõ ìîìåíòîâ, ïîñòðîåííîé îò äåéñòâèÿ åäèíè÷íîé ñèëû (*åñëè íàäî îïðåäåëèòü ïðîãèá*) èëè åäèíè÷íîãî ìîìåíòà (*åñëè íàäî îïðåäåëèòü óãîë ïîâîðîòà*);

EI– æåñòêîñòü áàëêè (*ïðîèçâåäåíèå ìîäóëÿ Þíãà íà îñåâîé ìîìåíò èíåðöèè ñå÷åíèÿ áàëêè îòíîñèòåëüíî íåéòðàëüíîé îñè*).

Ïðèìå÷àíèå: *åäèíè÷íàÿ ñèëà è åäèíè÷íûé ìîìåíò ïðèêëàäûâàþòñÿ â ñå÷åíèè, èñêîìûå ïåðåìåùåíèÿ êîòîðîãî òðåáóåòñÿ îïðåäåëèòü.*

Îïðåäåëèì ïðîãèá è óãîë ïîâîðîòà êîíöåâîãî ñå÷åíèÿ êîíñîëüíîé áàëêè (Ðèñ. 26).



Ðèñ 26

1. Ñòðîèì ýïþðó èçãèáàþùèõ ìîìåíòîâ îò ñèëû Ð.
2. Îïðåäåëÿåì ïëîùàäü ýïþðû: $w = –(1/2)Pl \cdot l = –Pl^2/2$.
3. Ñòðîèì ýïþðó îò ñèëû Ð = 1.
4. Îïðåäåëÿåì îðäèíàòó $Ì^0_Ñ$: $Ì^0_C = –2l/3$.
5. Îïðåäåëÿåì ïðîãèá êîíöåâîãî ñå÷åíèÿ

$$y = wM^0_Ñ/EI = 21(Pl^2)/(2 \cdot 3) = Pl^3/3EI$$ ì.

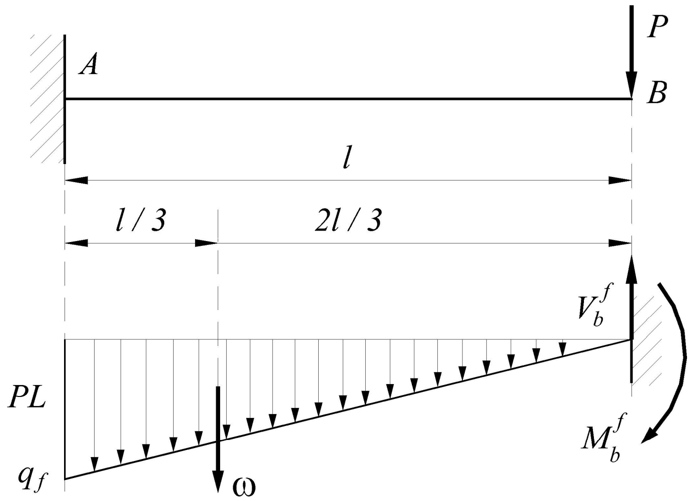
6. Ñòðîèì ýïþðó îò ìîìåíòà Ì = 1.
7. Îïðåäåëÿåì íà ýòîé ýïþðå îðäèíàòó $Ì^0_Ñ$: $Ì^0_Ñ = –1$.
8. Îïðåäåëÿåì óãîë ïîâîðîòà êîíöåâîãî ñå÷åíèÿ

$$q = wÌ^0_Ñ/EI = (Pl^2l)/2EI = Pl^2/2EI$$ ðàä.

Ïðèìå÷àíèå: *Åñëè ïåðåìåùåíèÿ ïîëîæèòåëüíû, òî èõ íàïðàâëåíèÿ äîëæíû ñîâïàäàòü ñ íàïðàâëåíèåì åäèíè÷íûõ íàãðóçîê.*

Ïðèìåðû ðàñ÷åòà

Ïðèìåð 1. Äëÿ êîíñîëüíîé áàëêè, ñ ïðèëîæåííîé ê íåé ñîñðåäîòî÷åííîé ñèëîé Ð, îïðåäåëèòü óãîë ïîâîðîòà è ïðîãèá ñå÷åíèÿ, ïðîõîäÿùåãî ÷åðåç òî÷êó Â (Ðèñ. 27).



Ðèñ. 27

Ð å ø å í è å. 1. Âû÷åð÷èâàåì ðàñ÷åòíóþ ñõåìó áàëêè.

2. Ñòðîèì ýïþðó èçãèáàþùèõ ìîìåíòîâ.

3. Òàê êàê ìîìåíò áóäåò îòðèöàòåëüíûì, ñòðåëêè ôèêòèâíîé ðàñïðåäåëåííîé íàãðóçêè íàïðàâëÿåì âíèç.

4. Ïðèíèìàåì ôèêòèâíóþ áàëêó.

5. Îïðåäåëÿåì âåëè÷èíó îïîðíûõ ðåàêöèé ôèêòèâíîé áàëêè:

S$Ì_{Â}$ = 0 = –M_B^f + w(2/3)1 = –$M_{Â}^f$ + (1/2)P×l×l×(2/3)×l; $Ì_{Â}^f$ = $Pl^3/3$;

SY = 0 = $V_{Â}^f$ – w = $V_{Â}^f$ – $Pl^2/2$; $V_{Â}^f$ = $Pl^2/2$.

6. Îïðåäåëÿåì âåëè÷èíû èçãèáàþùåãî ìîìåíòà è ïîïåðå÷íîé ñèëû â ñå÷åíèè, ïðîõîäÿùåì ÷åðåç òî÷êó Â:

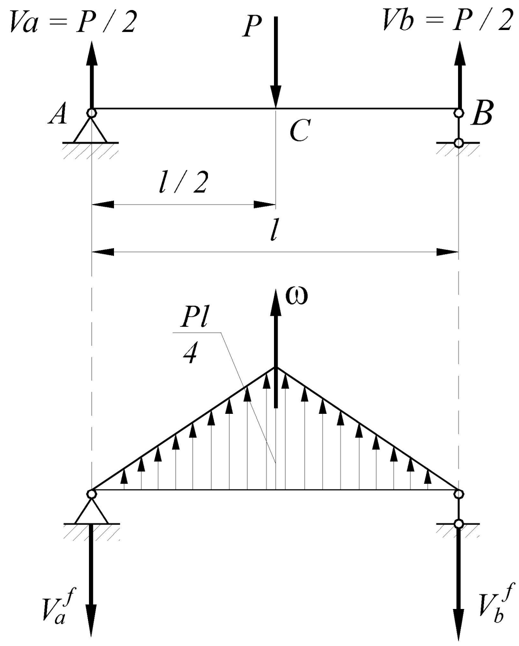
$Ì_{Â}^f$ = –$Pl^3/3$; $Q_{Â}^f$ = – $Pl^2/2$.

7. Îïðåäåëÿåì ïåðåìåùåíèÿ ïî ôîðìóëàì (7.8) è (7.9).

y_B = M_B^f/EI = –$3Pl^3/3EI$; Q_B = Q_B^f/EI = –$3Pl^2/2EI$.

Âèäíî, ÷òî ýòè îòâåòû èäåíòè÷íû ðåøåíèþ ìåòîäîì íåïîñðåäñòâåííîãî èíòåãðèðîâàíèÿ äèôôåðåíöèàëüíîãî óðàâíåíèÿ èçîãíóòîé îñè áàëêè.

Ïðèìåð 2. Äëÿ áàëêè, íàãðóæåííîé â ñåðåäèíå ïðîëåòà ñîñðåäîòî÷åííîé ñèëîé Ð, îïðåäåëèòü óãîë ïîâîðîòà ñå÷åíèÿ, ïðîõîäÿùåãî ÷åðåç òî÷êó Â, è ïðîãèá ñå÷åíèÿ, ïðîõîäÿùåãî ÷åðåç òî÷êó Ñ (Ðèñ. 28).



Ðèñ. 28

Ð å ø å í è å. 1. Âû÷åð÷èâàåì ðàñ÷åòíóþ ñõåìó áàëêè.

2. Ñòðîèì ýïþðó èçãèáàþùèõ ìîìåíòîâ.

3. Òàê êàê ìîìåíò áóäåò ïîëîæèòåëüíûì, íàïðàâëÿåì ñòðåëêè ôèêòèâíîé ðàñïðåäåëåííîé íàãðóçêè ââåðõ.

4. Ïðèìåíÿåì ôèêòèâíóþ áàëêó (Ðèñ. 28).

5. Îïðåäåëÿåì âåëè÷èíó ðåàêöèé ôèêòèâíîé áàëêè:

Òàê êàê íàãðóçêà ñèììåòðè÷íà $V_A^f = V_A^f = w/2 = (1/2)\ (Pl/4)(1/2) = Pl^2/16$.

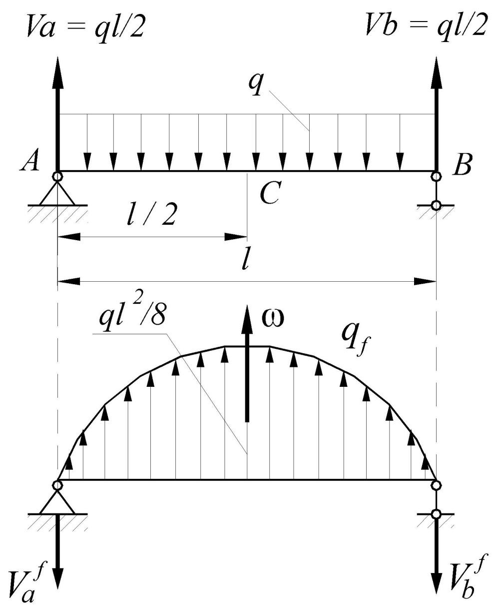
6. Îïðåäåëÿåì âåëè÷èíó ïîïåðå÷íîé ñèëû â ñå÷åíèè, ïðîõîäÿùåì ÷åðåç òî÷êó Â: $Q_f^B = V_B^f = Pl^2/16$;

7. Îïðåäåëÿåì âåëè÷èíó èçãèáàþùåãî ìîìåíòà â ñå÷åíèè, ïðîõîäÿùåì ÷åðåç òî÷êó Ñ: $Ì_f^C = –V_A^f(1/2) + (1/2)(Pl/4)(1/2)(1/3)(1/2) = – (Pl^3/32) + (Pl^3/96) =$
$= – Pl^3/48$.

8. Îïðåäåëÿåì ïåðåìåùåíèÿ Q_B è $ó_Ñ$: $Q_B = Q_f^B/EI = Pl^2/16EI$; $ó_Ñ = M_f^C/EI =$
$= –Pl^3/48EI$

.

Ïðèìåð 3. Äëÿ îäíîïðîëåòíîé áàëêè, íàãðóæåííîé ðàñïðåäåëåííîé íàãðóçêîé, îïðåäåëèòü ïðîãèá ó$_{Ñ}$ â ñåðåäèíå ïðîëåòà è óãîë ïîâîðîòà ñå÷åíèÿ Q$_{À}$, ïðîõîäÿùåãî ÷åðåç ëåâóþ îïîðó (Ðèñ. 29).



Ðèñ. 29

Ð å ø å í è å. 1. Âû÷åð÷èâàåì ðàñ÷åòíóþ ñõåìó áàëêè.

2. Ñòðîèì ýïþðó èçãèáàþùèõ ìîìåíòîâ.

3. Òàê êàê ìîìåíò ÿâëÿåòñÿ ïîëîæèòåëüíûì, íàïðàâëÿåì ñòðåëêè ôèêòèâíîé ðàñïðåäåëåííîé íàãðóçêè ââåðõ.

4. Ïðèíèìàåì ôèêòèâíóþ áàëêó ñì ðèñ.

5. Îïðåäåëÿåì âåëè÷èíó îïîðíûõ ðåàêöèé ôèêòèâíîé áàëêè:

$$V_A^f = V_A^f = w/2.$$

Îïðåäåëÿåì ïëîùàäü w èíòåãðèðîâàíèåì óðàâíåíèÿ èçãèáàþùèõ ìîìåíòîâ: $M = (ql/2)x - qx^2/2$;

$w = \int_0^1 (dl/2)x - qx^2/2]dx = ql^3/4 - ql^3/6 = ql^3/12$, òîãäà $V_A^f = V_A^f = ql^3/24$.

6. Îïðåäåëÿåì ïîïåðå÷íóþ ñèëó â ñå÷åíèè, ïðîõîäÿùåì ÷åðåç òî÷êó À.

$$Q_f^A = -V_A^f = -ql^3/24.$$

7. Îïðåäåëÿåì âåëè÷èíó èçãèáàþùåãî ìîìåíòà â ñå÷åíèè, ïðîõîäÿùåì ÷åðåç òî÷êó Ñ ôèêòèâíîé áàëêè (Ðèñ. 7.30): $M_f^C = -(1/2)V_A^f \times + (w/2)(1/2 - x_C)$; íàéäåì êîîðäèíàòó öåíòðà òÿæåñòè $x_C = S_Y/F$ [ñì. ôîðìóëó (2.4)].

Â íàøåì ñëó÷àå $F = w/2 = ql^3/24$, à

$S_Y = \int_0^{l/2} (\text{Ì}\,dx)x = \int_0^{l/2} [(ql/2)x - qx^2/2]x\,dx = ql^4/48 - ql^4/128 = 5ql^4/384$, òîãäà

$x_C = 5ql^4/384 : ql^3/24 = 5l/16$, ñëåäîâàòåëüíî
$M_f^C = -(ql^3/24)(l/2) + (ql^3/24)(l/2 - 5l/16) = ql^4/48 + 3ql^4/384 = 5ql^4/384$.

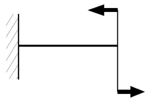
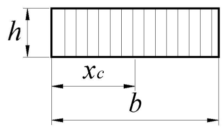
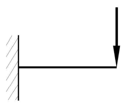
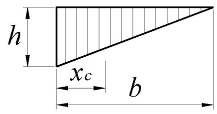
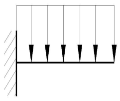
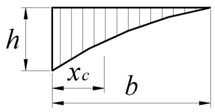
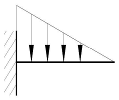
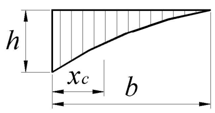
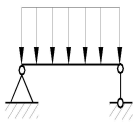
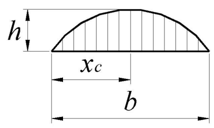
8. Îïðåäåëÿåì óãîë ïîâîðîòà $Q_{\text{À}}$ è ïðîãèá $ó_{\text{Ñ}}$:

$$Q_{\text{À}} = Q_f^A/EI = -ql^3/24EI;\quad y_C = -5ql^4 384EI.$$

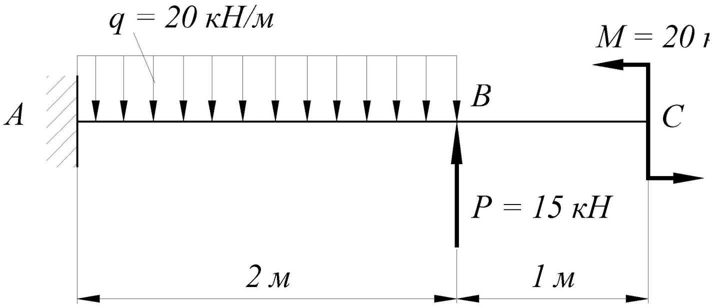
Èç ïðèâåäåííûõ ïðèìåðîâ âèäíî, ÷òî êîíå÷íûå ðåçóëüòàòû ðàñ÷åòîâ ñîâïàäàþò ñ îòâåòàìè çàäà÷, ðåøåííûõ ðàíåå ìåòîäîì íåïîñðåäñòâåííîãî èíòåãðèðîâàíèÿ äèôôåðåíöèàëüíîãî óðàâíåíèÿ èçîãíóòîé îñè.

Èç ïðèìåðà (3) âèäíî, ÷òî âû÷èñëåíèå ïëîùàäåé ýïþð èçãèáàþùèõ ìîìåíòîâ è êîîðäèíàò èõ öåíòðîâ òÿæåñòè èíòåãðèðîâàíèåì ñâÿçàíî ñ îïðåäåëåííûìè ñëîæíîñòÿìè, ïîýòîìó â ïðàêòè÷åñêèõ ðàñ÷åòàõ ïðè ñëîæíîì íàãðóæåíèè áàëêè ñòðîÿò *"ðàññëîåííûå"* ýïþðû. Ïðè ýòîì âíà÷àëå ñòðîÿò ýïþðû îò êàæäîé èç âíåøíèõ ñèë, ïðèëîæåííûõ ê áàëêå, *â îòäåëüíîñòè,* à ïîòîì ñêëàäûâàþò èõ âñå âìåñòå íà îñü ýïþðû. Òîãäà ïðèõîäèòñÿ èìåòü äåëî òîëüêî ñ ïëîùàäÿìè è êîîðäèíàòàìè ýïþð, êîòîðûå ïðèâåäåíû â òàáëèöå 2.

Òàáëèöà 2

Ñõåìà íàãðóæåíèÿ	Ýïþðà Ì	Ïëîùàäü w	Êîîðäèíàòà $õ_{\text{Ñ}}$
		bh	b/2
		bh/2	b/3
		bh/3	b/4
		bh/4	b/5
		2bh/3	b/2

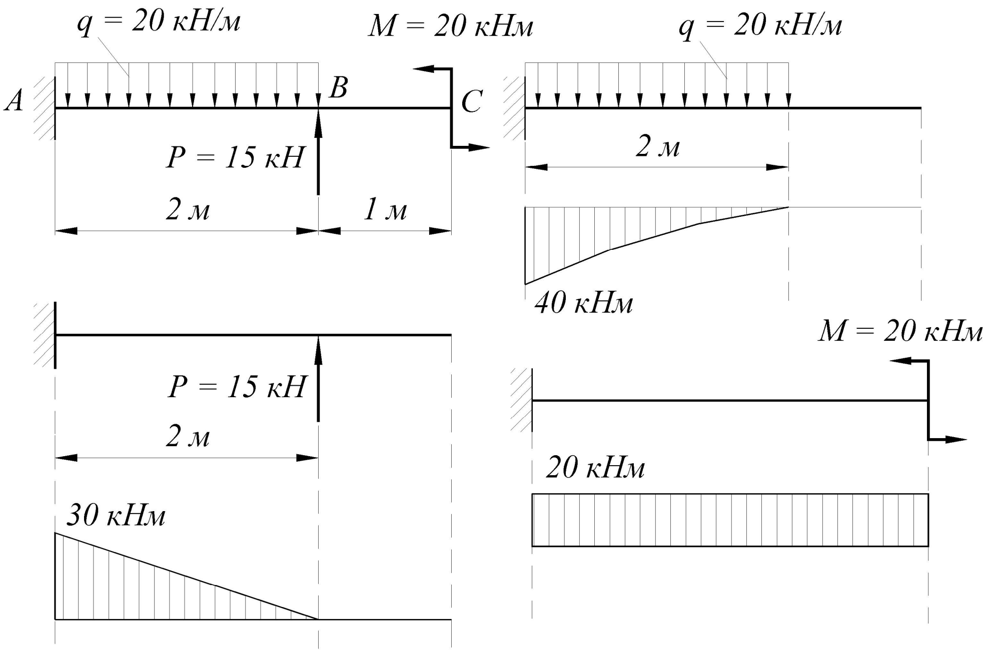
Ïðèìåð 4. Äëÿ êîíñîëüíîé áàëêè îïðåäåëèòü óãîë ïîâîðîòà ñå÷åíèÿ, ïðîõîäÿùåãî ÷åðåç òî÷êó Ñ, è ïðîãèá ñå÷åíèÿ, ïðîõîäÿùåãî ÷åðåç òî÷êó Â (Ðèñ.31).



Ðèñ.31

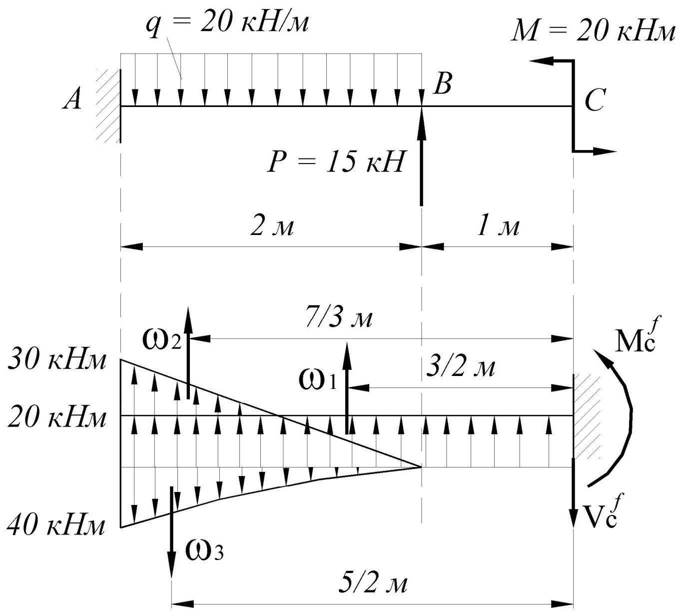
Ð å ø å í è å. 1. Âû÷åð÷èâàåì ðàñ÷åòíóþ ñõåìó áàëêè (Ðèñ. 31).

2. Ñòðîèì ýïþðû èçãèáàþùèõ ìîìåíòîâ îò êàæäîé èç ñèë, ïðèëîæåííûõ ê áàëêå (Ðèñ. 32, à), â îòäåëüíîñòè (Ðèñ. 32, á, â, ã).



Ðèñ. 32

3. Ñâîäèì âñå òðè ýïþðû íà îñü áàëêè (Ðèñ. 33). Íà ýïþðàõ ñ ïîëîæèòåëüíûìè çíà÷åíèÿìè ìîìåíòîâ ñòðåëêè ôèêòèâíîé ðàñïðåäåëåííîé íàãðóçêè íàïðàâëÿåì ââåðõ, à íà ýïþðå ñ îòðèöàòåëüíûìè çíà÷åíèÿìè ìîìåíòîâ – âíèç.



Ðèñ. 33

4. Ïðèíèìàåì ôèêòèâíóþ áàëêó.

5. Îïðåäåëÿåì âåëè÷èíó îïîðíûõ ðåàêöèé ôèêòèâíîé áàëêè:

SÌ$_{Ñ}$ = 0 = M$_{Ñ}^{f}$ + w$_{3}$(5/2) – w$_{2}$(7/3) – w$_{1}$(3/2), èëè

M$_{C}^{f}$ = + (1/2)·30·2·(7/3) – (1/3)×40·2·(5/2) + 20·3·(3/2) » 93,33 êÍì^{3};

SY = 0 = –V$_{C}^{f}$ + w$_{1}$ + w$_{2}$ – w$_{3}$, èëè V$_{Ñ}^{f}$ = 20·3 + (1/2)·30·2 – (1/3)·40·2 » 63,3 êÍì^{2}.

6. Îïðåäåëÿåì âåëè÷èíó ïîïåðå÷íîé ñèëû â ñå÷åíèè ôèêòèâíîé áàëêè, ïðîõîäÿùåì ÷åðåç òî÷êó Ñ, è èçãèáàþùèé ìîìåíò M$_{f}^{B}$.

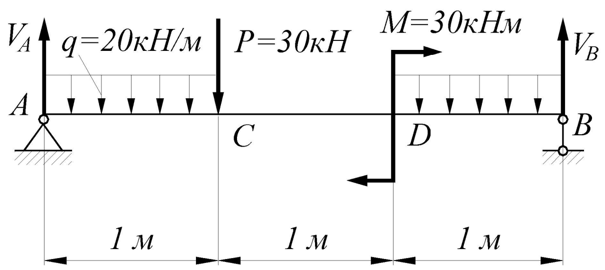
Q$_{f}^{C}$ = V$_{C}^{f}$ = 63,3 êÍì^{2}; M$_{f}^{B}$ = M$_{C}^{f}$ + 20·1×0,5 – V$_{C}^{f}$×1 = 93,3 + 10 – 63,3 = 40,0 êÍì^{3}.

7. Îïðåäåëÿåì Q$_{Ñ}$ è ó$_{Â}$, åñëè áàëêà èçãîòîâëåíà èç äâóòàâðà ¹ 30, ó êîòîðîãî îñåâîé ìîìåíò èíåðöèè îòíîñèòåëüíî íåéòðàëüíîé îñè I = 7080 ñì^{4}.

Q$_{Ñ}$ = Q$_{f}^{C}$ / EI = 63,3/(2×10^{8}×7080×10^{-8}) = 0,0044 ðàä = 0,25°;

y$_{C}$ = M$_{f}^{B}$ / EI = 40/(2×10^{8}×7080×10^{-8}) = 0,0028 ì = 2,8 ìì.

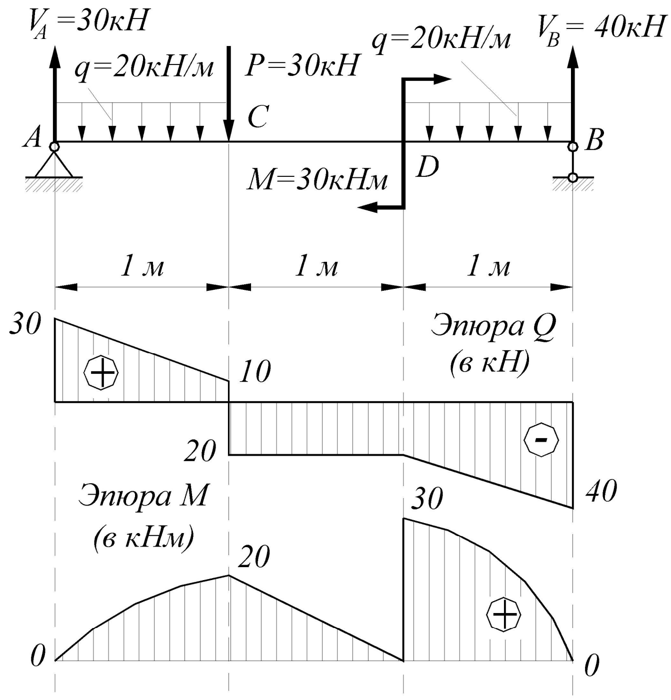
Ïðèìåð 5. Äëÿ çàäàííîé áàëêè (Ðèñ. 34) ïîñòðîèòü ýïþðû ïîïåðå÷íûõ ñèë è èçãèáàþùèõ ìîìåíòîâ.



Ðèñ. 34

Ïîäîáðàòü êâàäðàòíîå ïîïåðå÷íîå ñå÷åíèå èç äåðåâà, ïîëüçóÿñü óñëîâèåì ïðî÷íîñòè ïî íîðìàëüíûì íàïðÿæåíèÿìè (R = 9 ÌÏà) è ïîñòðîèòü èçîãíóòóþ îñü áàëêè ñ ïîìîùüþ ìåòîäà íà÷àëüíûõ ïàðàìåòðîâ. Îïðåäåëèòü äëÿ ýòîãî óãëû ïîâîðîòà ñå÷åíèé, ïðîõîäÿùèõ ÷åðåç îïîðû, è ïðîãèáû ñå÷åíèé, ïðîõîäÿùèõ ÷åðåç òî÷êè Ñ è D.

Ð å ø å í è å. 1. Ñ÷èòàÿ îïîðíûå ðåàêöèè îïðåäåëåííûìè, ñòðîèì "ýïþðû èçãèáàþùèõ ìîìåíòîâ (Ðèñ. 35).



Ðèñ. 35

Ïðèìå÷àíèå: *Âî âñåõ ïîñëåäóþùèõ çàäà÷àõ ñàìîñòîÿòåëüíî îïðåäåëèòå îïîðíûå ðåàêöèè è ïîñòðîèòå ýïþðû Q è Ì.*

2. Ïîäáèðàåì ïîïåðå÷íîå ñå÷åíèå áàëêè, ïîëüçóÿñü óñëîâèåì ïðî÷íîñòè ïî íîðìàëüíûì íàïðÿæåíèÿì:

s = Ì$_{max}$ /W £ R, îòñþäà W = Ì$_{max}$ /R = $30 \cdot 10^{-3}$ /9 = 0,003333 ì^3 = 3333 ñì^3.

Òàê êàê äëÿ êâàäðàòíîãî ñå÷åíèÿ W = à^3 /6 [ñì. ôîðìóëó (6.6)], òî

à = ^{3}Ö6W = ^{3}Ö 6×3333 » 27ñì.

3. Îïðåäåëÿåì ïåðåìåùåíèÿ áàëêè.

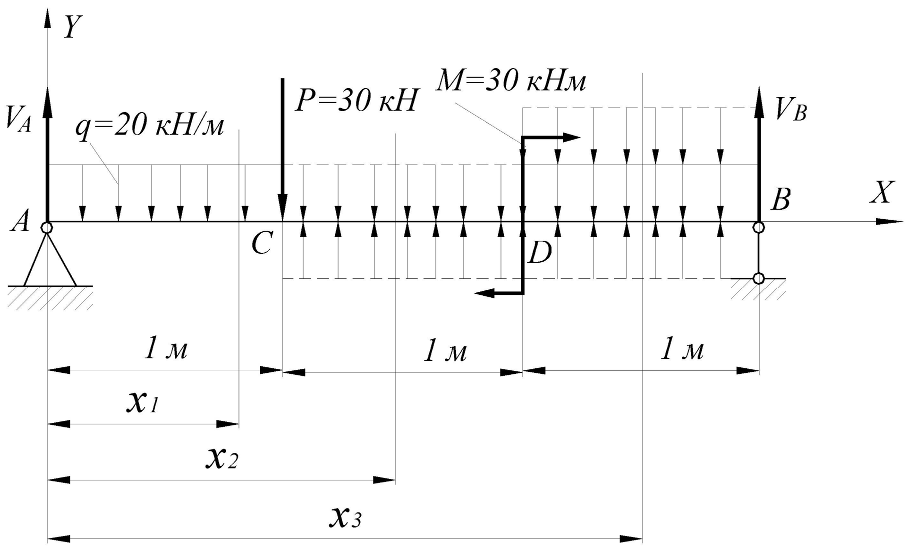
3.1. Îïðåäåëÿåì âåëè÷èíó îñåâîãî ìîìåíòà èíåðöèè ñå÷åíèÿ áàëêè îòíîñèòåëüíî íåéòðàëüíîé îñè:

I$_X$ = à^4 /12 = 27^4 /12 = 44286,75 ñì^4 = $44286{,}75 \cdot 10^{-8}$ ì^4.

3.2. Îïðåäåëÿåì æåñòêîñòü áàëêè (Å$_{äåð}$= $1 \cdot 10^4$ ÌÏà):

EI = $44286{,}75 \times 10^{-8} \times 10^4 \times 10^3$ » 4429 êÍì^2.

3.3. Ïðèíèìàåì íà÷àëî êîîðäèíàò äëÿ îïðåäåëåíèÿ ïåðåìåùåíèé è ïðîâîäèì ïîïåðå÷íûå ñå÷åíèÿ íà êàæäîì ó÷àñòêå áàëêè (Ðèñ. 36).



Ðèñ. 36

3.4. Çàïèñûâàåì îáîáùåííîå óðàâíåíèå èçîãíóòîé îñè áàëêè äëÿ ïåðâîãî ñå÷åíèÿ:

$$EIy_1 = EIy_0 + EIq_0x_1 + V_A(x_1^3/6) - q(x_1^4/24), \qquad (à)$$

òàê êàê â îïîðå ïðîãèá ó$_A$ = 0, ìîæåì çàïèñàòü, ÷òî EIy$_À$(ïðè x_1=0) = 0 = EIy$_0$.

3.5. Çàïèñûâàåì îáîáùåííîå óðàâíåíèå èçîãíóòîé îñè äëÿ òðåòüåãî ñå÷åíèÿ:

$$EIy_3 = EIq_0x_3 + V_A(x_3^3/6) - q(x_3^4/24) - P(x_3 - 1)^3/6 + M(x_3 - 2)^2/2 + q\,(x_3 - 1)^4/24 - q\,(x_3 - 2)^4/24. \qquad (b)$$

Òàê êàê ïðîãèá y_B = 0, ïîäñòàâèì â ïîëó÷åííîå óðàâíåíèå x_3 = 3 ì.

EIy_B = 0 = EIq_0×3 + 55, îòñþäà EIq_0×Ç = –55, a EIq_0 = –18,33 êÍì^2, òîãäà q_0 = q_A = –18,33 / EI = –18,33 /4429 = 0,0041 ðàä = 0,2373^0.

3.6. Îïðåäåëÿåì ïðîãèá ó$_ñ$. Ïîäñòàâëÿåì äëÿ ýòîãî â óðàâíåíèå (a) x_1 = 1ì:

EIy_c = –18,33·1 + 30(1/6) – 20(1/24) = –14,17 êÍì^3;

ó$_Ñ$ = –14,47/4429 = – 0,003198 ì = –0,32 ñì.

3.7. Îïðåäåëÿåì ïðîãèá ó$_D$. Çàïèñûâàåì îáîáùåííîå óðàâíåíèå èçîãíóòîé îñè äëÿ âòîðîãî ñå÷åíèÿ è ïîäñòàâëÿåì â íåãî õ$_2$ = 2ì:

EIy_2= EI$q_0$$x_2$ +V$_À$(x_2^3/6) – q(x_2^4/24) + q(x_2 – 1)^4/24 – P(x_2 – 1)^3/6;

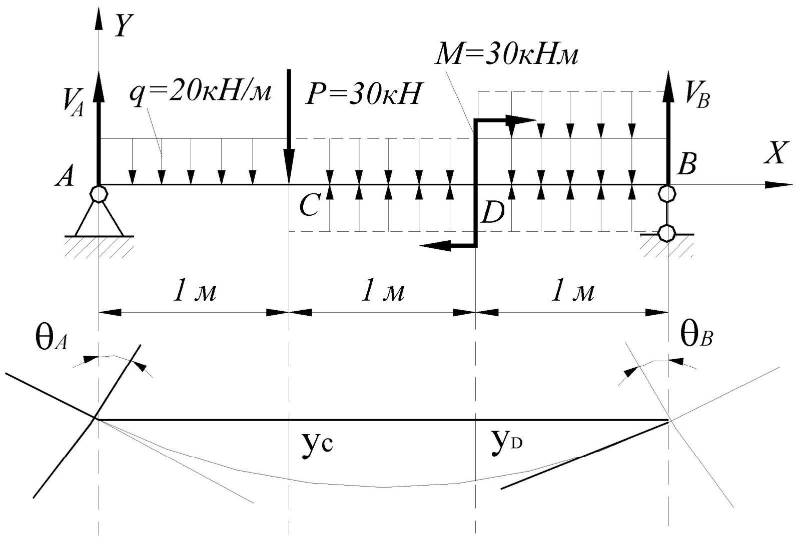
EIy_D (ïðè x_2=2 ì) = –14,17 êÍì^3; ó$_D$ = –14,17/4429 = – 0,003198 ì = 0,32 ñì .

3.8. Îïðåäåëÿåì óãîë ïîâîðîòà $q_â$. Çàïèñûâàåì äëÿ ýòîãî îáîáùåííîå óðàâíåíèå èçîãíóòîé îñè äëÿ òðåòüåãî ñå÷åíèÿ è ïîäñòàâëÿåì â íåãî õ$_3$ = 3ì:

EIq_3 = EIq_0+V$_À$(x_3^2/6)–q(x_3^3/6) + q(x_3 – 1)^3/6 – P(x_3–1)^2/2+M(x_3 – 2)–q(x_3 – 2)^3/6;

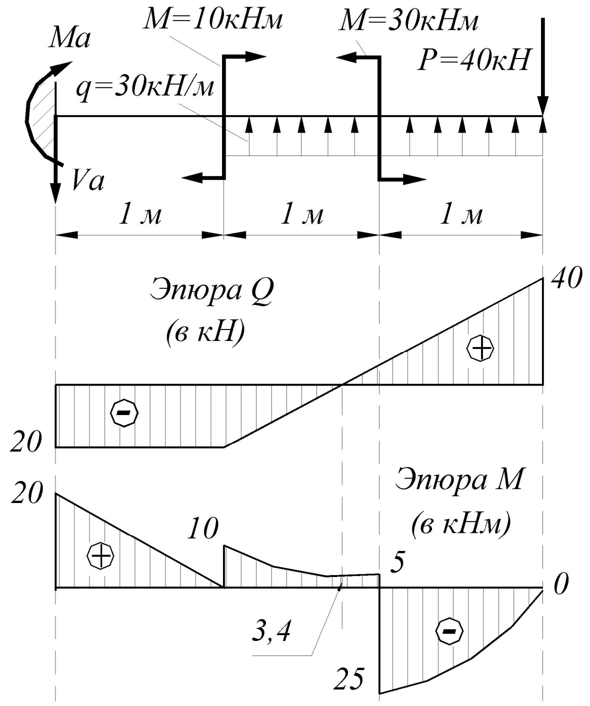
ÅI$q_Â$ (ïðè õ$_3$ = 3 ì) = 20 êÍì^2; $q_Â$ = 20/4429 = 0,0045 ðàä = 0,258°.

3.9. Ïî ïîëó÷åííûì äàííûì ñòðîèì èçîãíóòóþ îñü áàëêè (Ðèñ. 37).



Ðèñ. 37

Ïðèìåð 6. Äëÿ çàäàííîé áàëêè (Ðèñ. 38) ïîñòðîèòü ýïþðû ïîïåðå÷íûõ ñèë è èçãèáàþùèõ ìîìåíòîâ, ïîäîáðàòü êðóãëîå ïîïåðå÷íîå ñå÷åíèå èç äåðåâà, ïîëüçóÿñü óñëîâèåì ïðî÷íîñòè ïî íîðìàëüíûì íàïðÿæåíèÿì (R = 9 ÌÏà) è ïîñòðîèòü èçîãíóòóþ îñü áàëêè ñ ïîìîùüþ ìåòîäà íà÷àëüíûõ ïàðàìåòðîâ. Îïðåäåëèòü äëÿ ýòîãî $q_{Â}$ è q_{D}, à òàêæå ïðîãèáû $ó_{B}$, $ó_{C}$ è $ó_{Î}$.



Ðèñ. 38

Ð å ø å í è å. 1. Ñòðîèì ýïþðû ïîïåðå÷íûõ ñèë è èçãèáàþùèõ ìîìåíòîâ (Ðèñ. 38).

2. Ïîäáèðàåì ïîïåðå÷íîå ñå÷åíèå áàëêè, ïîëüçóÿñü óñëîâèåì ïðî÷íîñòè ïî íîðìàëüíûì íàïðÿæåíèÿì [ñì. ôîðìóëó (6.5)]:

s = M_{max} / W £ R, îòñþäà W = M_{max} / R = $25 \cdot 10^{-3}$ / 9 = 0,0028 $ì^{-3}$ = 2800 $ñì^{3}$.

Òàê êàê äëÿ êðóãëîãî ñå÷åíèÿ W = $0,1d^{3}$, òî

d = ³ÖW / 0,1 = ³Ö28000 = 30,4 ñì » 30ñì.

3. Îïðåäåëÿåì ïåðåìåùåíèÿ áàëêè.

3.1. Îïðåäåëÿåì âåëè÷èíó îñåâîãî ìîìåíòà èíåðöèè áàëêè îòíîñèòåëüíî íåéòðàëüíîé îñè:

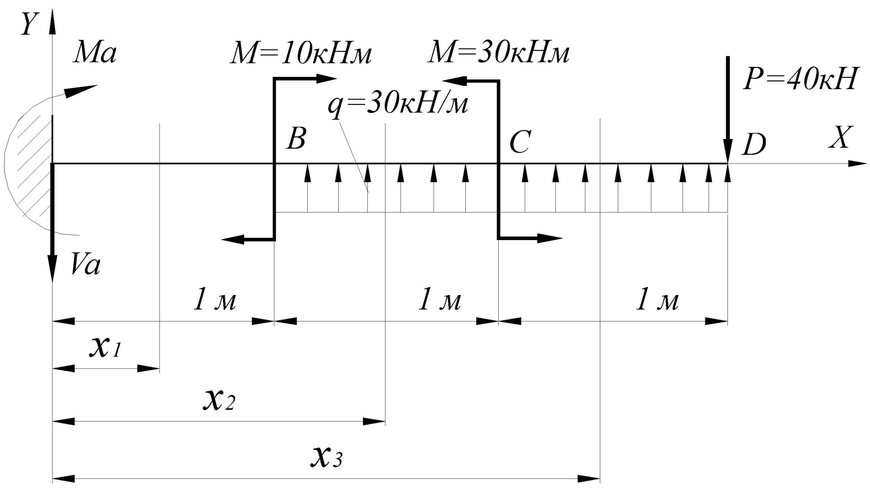
I = pd^{4} / 64 = $3,14 \cdot 30^{4}$ / 64 = 39740,63 $ñì^{4}$ » $39741 \cdot 10^{-8}$ $ì^{4}$.

3.2. Îïðåäåëÿåì æåñòêîñòü áàëêè [$Å_{äåð}$ = 10^{4} ÌÏà]:

EI = $10^{4} \cdot 10^{3} \cdot 39741 \cdot 10^{-8}$ » 3974 $êÍì^{2}$.

3.3. Ïðèíèìàåì íà÷àëî êîîðäèíàò â çàäåëêå áàëêè, òàê êàê íà÷àëüíûå ïàðàìåòðû $q_A = q_0 = 0$ è $ó_A = ó_0 = 0$ è, ñëåäîâàòåëüíî, îïðåäåëÿòü èõ íå íàäî.

Ïðîâîäèì íà êàæäîì ó÷àñòêå áàëêè ïîïåðå÷íîå ñå÷åíèå (Ðèñ. 39).



Ðèñ. 39

3.4. Çàïèñûâàåì îáîáùåííûå óðàâíåíèÿ èçîãíóòîé îñè áàëêè äëÿ ïåðâîãî ñå÷åíèÿ è ïîäñòàâëÿåì â íèõ $x_1 = 1$ ì:

$EIy_1 = M_A(x_1^2/2) - V_A(x_1^3/6)$; $EIq_1 = Ì_A x_1 - V_A(x_1^2/2)$;

$EIó_Â$ (ïðè õ = 1 ì) = 6,67 êÍì^3, òîãäà $ó_Â$ = 6,67/3974 = 0,0017 ì = 0,17 ñì;

$ÅIq_Â$ (ïðè x_1 = 1 ì) = 10 êÍì^2, òîãäà $q_Â$ = 10/3974 = 0,0025 ðàä = 0,14^0.

3.5. Çàïèñûâàåì îáîáùåííîå óðàâíåíèå èçîãíóòîé îñè áàëêè äëÿ âòîðîãî ñå÷åíèÿ è ïîäñòàâëÿåì â íåãî $x_2 = 2$ì:

$EIy_2 = M_À x_2^2/2 - V_À x_2^3/6 + Ì(x_2 - 1)^2/2 + q(x_2 - 1)^4/24$;

$EIy_Ñ$ (ïðè x_2 = 2 ì) = 19,58 êÍì^3, òîãäà $ó_C$ = 19,58/3974 = 0,0049 ì = 0,49 ñì.

3.6. Çàïèñûâàåì îáîáùåííûå óðàâíåíèÿ èçîãíóòîé îñè áàëêè äëÿ òðåòüåãî ñå÷åíèÿ è ïîäñòàâëÿåì â íèõ $õ_3$ = 3 ì:

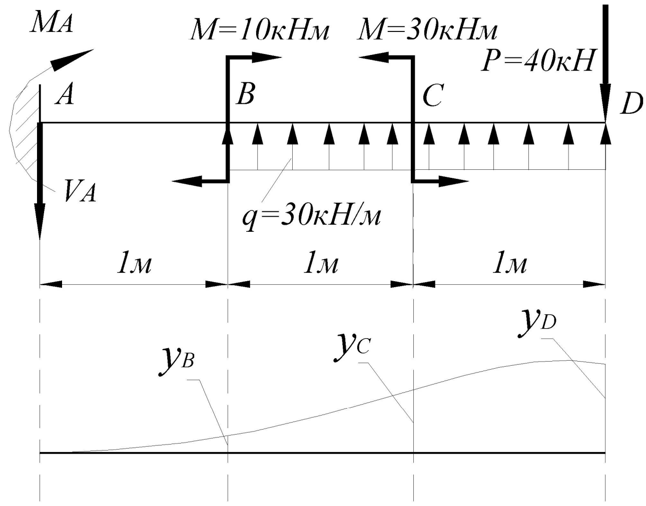
$EIy_3 = M_À x_3^2/2 - V_À x_3^3/6 + Ì_1(x_3 - 1)^2/2 + q(x_3 - 1)^4/24 - M_2(x_3 - 2)^2/2$;

$EIq_3 = Ì_À x_3 - V_À x_3^2/2 + Ì_1(x_3 - 1) + q(x_3 - 1)^3/6 - M_2(x_3 - 2)$;

EIy_D (npu $õ_3$ = 3 ì) = 25 êÍì^3, òîãäà $ó_D$ = 25/3974 = 0,0063 ì = 0,63 ñì;

EIq_D (ïðè x_3 = 3 ì) = 0 êÍì^2, òîãäà q_D = 0 ðàä = 0°.

3.7. Ïî ïîëó÷åííûì äàííûì ñòðîèì èçîãíóòóþ îñü áàëêè (Ðèñ. 40).



Ðèñ. 40

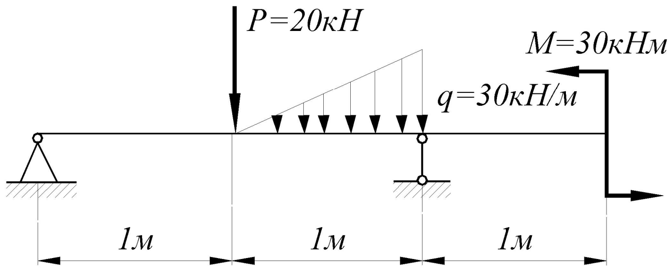
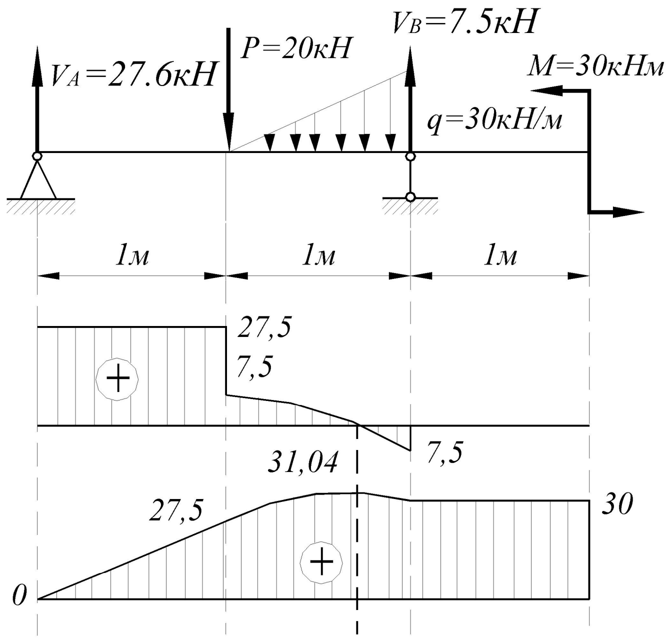


Рис. 41

Ïðèìåð 7. Äëÿ çàäàííîé áàëêè (Ðèñ. 41) ïîñòðîèòü ýïþðû ïîïåðå÷íûõ ñèë è èçãèáàþùèõ ìîìåíòîâ, ïîäîáðàòü äâóòàâðîâîå ïîïåðå÷íîå ñå÷åíèå, åñëè ðàñ÷åòíîå ñîïðîòèâëåíèå R = 160 ÌÏà. Ïîñòðîèòü èçîãíóòóþ îñü áàëêè, ïîëüçóÿñü ìåòîäîì íà÷àëüíûõ ïàðàìåòðîâ. Äëÿ ýòîãî îïðåäåëèòü: óãëû ïîâîðîòà $q_{\text{À}}$ è $q_{\text{Â}}$; ïðîãèáû $ó_C$ è $ó_D$.

Ð å ø å í è å. 1. Ñòðîèì ýïþðû ïîïåðå÷íûõ ñèë è èçãèáàþùèõ ìîìåíòîâ (Ðèñ. 42).



Ðèñ. 42

2. Ïîäáèðàåì äâóòàâðîâîå ïîïåðå÷íîå ñå÷åíèå áàëêè, ïîëüçóÿñü óñëîâèåì ïðî÷íîñòè ïî íîðìàëüíûì íàïðÿæåíèÿì:

$s = M_{max} / W < R$, îòñþäà $W = M_{max} / R = 31{,}04{\times}10^{-3} / 160 = 1940{\times}10^{7}$ $ì^3$ = 194 $ñì^3$.

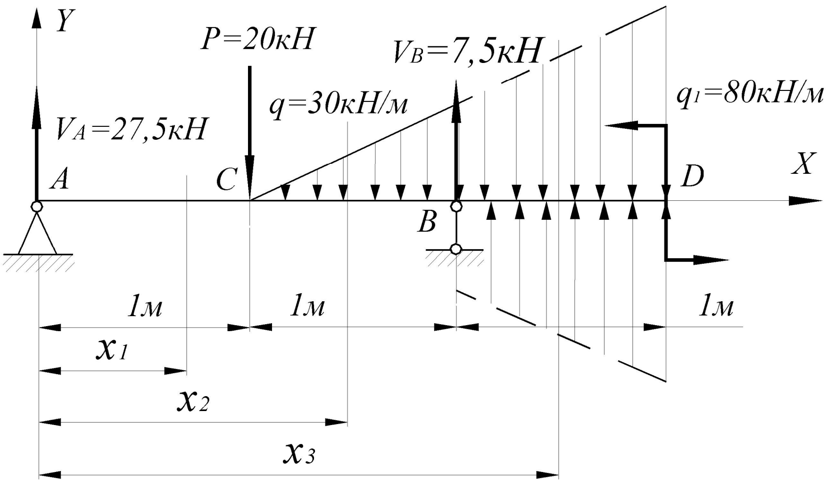
Èç ÃÎÑÒ 8239–89 âûáèðàåì äâóòàâð ¹ 20à ñ $W_{õ}$ = 203 $ñì^3$.

3. Îïðåäåëÿåì ïåðåìåùåíèÿ áàëêè.

3.1. Èç ÃÎÑÒ 8239–89 âûïèñûâàåì äëÿ äâóòàâðà ¹ 20à $I_{\text{õ}} = 2030$ ñì^4.

3.2. Îïðåäåëÿåì æåñòêîñòü áàëêè ($E_{\text{ñòàëè}} = 2 \times 10^8$ êÏà):
$EI = 2 \times 10^8 \times 2030 \times 10^{-8} = 4060$ êÍì^2.

3.3. Ïðèíèìàåì íà÷àëî êîîðäèíàò íà ëåâîì êîíöå áàëêè (Ðèñ. 43).



Ïðîâîäèì êîîðäèíàòíûå îñè ÷åðåç òî÷êó À è ïîïåðå÷íûå ñå÷åíèÿ íà êàæäîì ó÷àñòêå áàëêè. Òðåóãîëüíóþ ðàñïðåäåëåííóþ íàãðóçêó ïðîäîëæàåì äî êîíöà áàëêè, ïðîòèâîïîëîæíóþ íà÷àëó êîîðäèíàò, è óðàâíîâåøèâàåì åå òàêîé æå íàãðóçêîé, ïðèëîæåííîé ñíèçó ââåðõ íà òðåòüåì ó÷àñòêå.

3.4. Îïðåäåëÿåì âåëè÷èíó íà÷àëüíûõ ïàðàìåòðîâ. Äëÿ ýòîãî çàïèñûâàåì îáîáùåííûå óðàâíåíèÿ ïðîãèáîâ äëÿ ïåðâîãî è âòîðîãî ñå÷åíèé.

$EIy_1 = EIy_0 + EIq_0 x_1 + V_{\text{À}} x_1^3/6$;

$EIy_2 = EIy_0 + EIq_0 x_2 + V_{\text{À}} x_2^3/6 - P(x_2 - 1)^3/6 - q(x_2 - 1)^5/120$.

Î÷åâèäíî, ÷òî ïðîãèáû â òî÷êàõ À è Â áóäóò ðàâíû íóëþ. Ïîýòîìó ïîäñòàâèì â ïåðâîå óðàâíåíèå $x_1 = 0$, à âî âòîðîå – $x_1 = 2$ ì.

EIy_A (ïðè $x_1 = 0$) $= 0 = EIy_0$; $EIy_{\text{Â}}$ (ïðè $x_2 = 2$ ì) $= EIq_0 \times 2 + 33,08$ èëè

$EIq_0 = -16,54$ êÍì^2, òîãäà $q_0 = q_{\text{À}} = -1654/4060 = -0,0041$ ðàä $= 0,235°$.

3.5. Îïðåäåëÿåì óãîë ïîâîðîòà $q_{\text{Â}}$. Ñîñòàâëÿåì äëÿ ýòîãî óðàâíåíèå óãëà ïîâîðîòà äëÿ âòîðîãî ñå÷åíèÿ è ïîäñòàâëÿåì â íåãî $\text{õ}_2 = 2$ ì:

$EIq_2 = EIq_0 + V_{\text{À}} x_2^2/2 - \text{Ð}(x_2 - 1)^2/2 - q(x_2 - 1)^4/120$;

EIq_B (ïðè $x_2 = 2$ ì) $= 27,21$ êÍì^2, òîãäà $q_{\text{Â}} = 27,21/4060 = 0,0067$ ðàä $= 0,384°$.

3.6. Îïðåäåëÿåì ïðîãèá $\text{ó}_{\text{ñ}}$. Ïîäñòàâëÿåì ñ ýòîé öåëüþ $x_1 = 1$ ì â óðàâíåíèå ïðîãèáîâ äëÿ ïåðâîãî ñå÷åíèÿ;

EIy_C (ïðè $x_1 = 1$ ì) $= -11,96$ êÍì^3, òîãäà $\text{ó}_C = -11,96/4060 = -0,0029$ ì =
= – 0,29 ñì

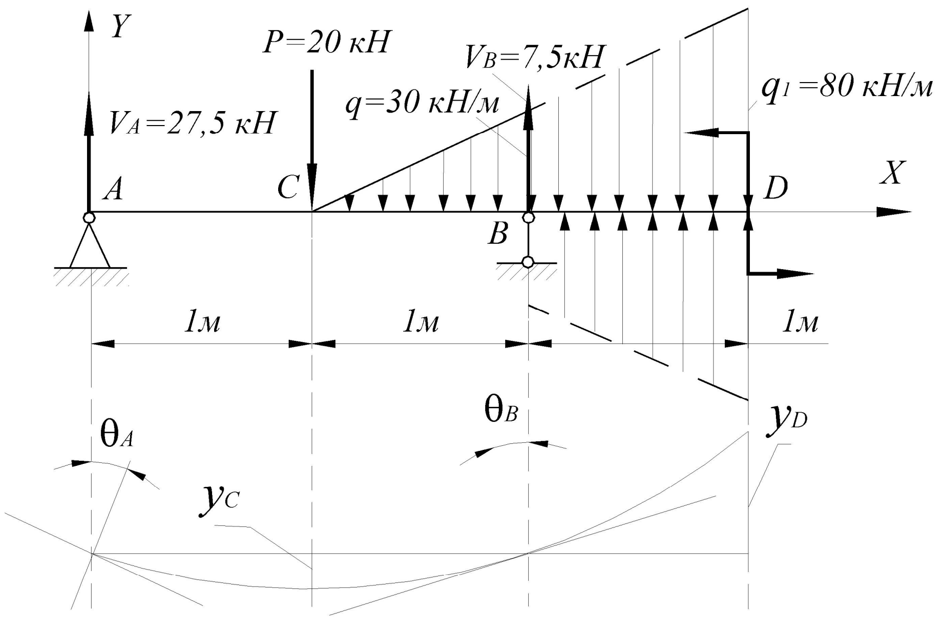
3.7. Îïðåäåëÿåì ïðîãèá ó_0. Ñîñòàâëÿåì óðàâíåíèå ïðîãèáîâ äëÿ òðåòüåãî ñå÷åíèÿ è ïîäñòàâëÿåì â íåãî $\text{õ}_3 = \text{Ç}$ ì:

$EIy_3 = EIq_0 x_3 + V_{\text{À}} x_3^3/6 - P(x_3 - 1)^3/6 - q(x_3 - 1)^5/120 + V_B(x_3 - 2)^4/24 + q(x_3 - 2)^4/24 + q(x_3 - 2)^5/120$; ÅIó_0 (ïðè $\text{õ}_3 = 3$ ì) $= 42,21$ êÍì^3, òîãäà

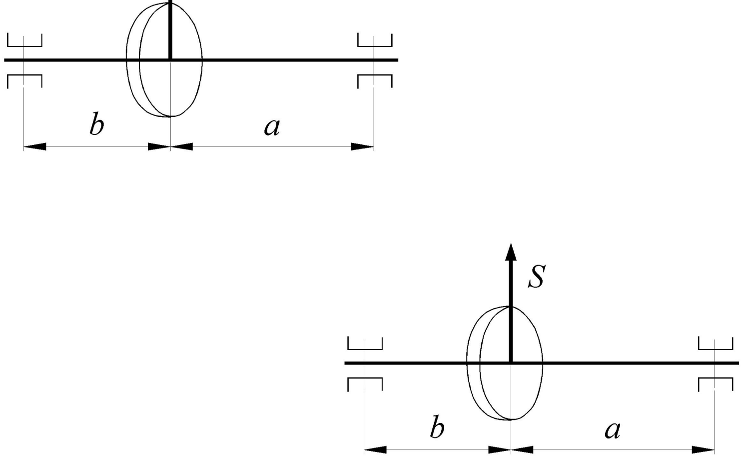
$y_D = 42,21/4060 = 0,0104$ м $= 1,04$ см.

3.8. Ïî ïîëó÷åííûì äàííûì ñòðîèì èçîãíóòóþ îñü áàëêè (Ðèñ..44).

Ïðèìå÷àíèå: âî âñåõ çàäà÷àõ, ðåøåííûõ ñ ïîìîùüþ ìåòîäà íà÷àëüíûõ ïàðàìåòðîâ, ïðè ñîñòàâëåíèè îáîáùåííûõ óðàâíåíèé èçîãíóòîé îñè èñïîëüçîâàëèñü ôîðìóëû (6) è (7).



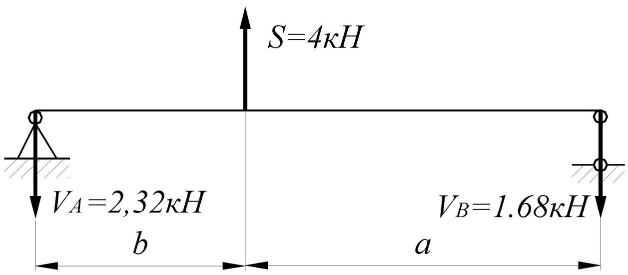
Ðèñ..44



Ðèñ. 45

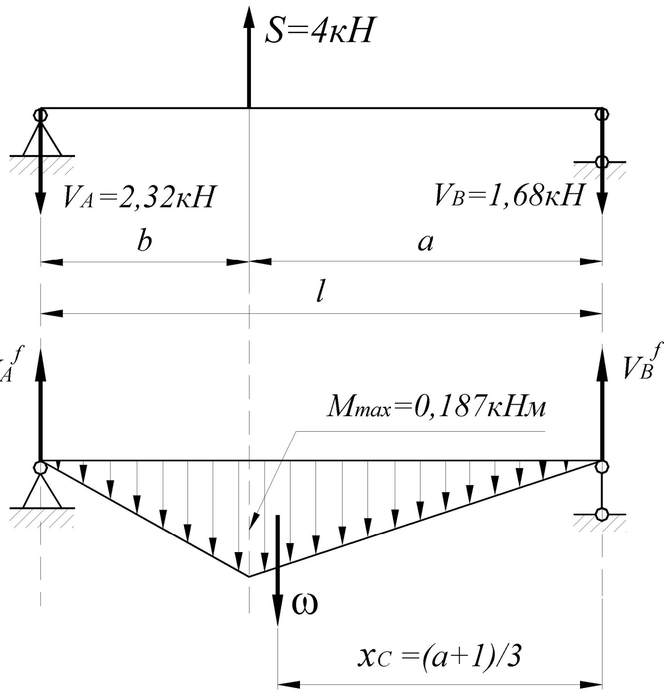
Ïðèìåð 8. Ïðîâåðèòü æåñòêîñòü âàëà ðåäóêòîðà), åñëè ïðîãèá â ïëîñêîñòè, ïðîõîäÿùåé ÷åðåç îñè âàëîâ â ìåñòå óñòàíîâêè øåñòåðíè íå äîëæåí ïðåâûøàòü [y] = 0,1 ìì. Äèàìåòð âàëà d = 32 ìì, ðàññòîÿíèå îò ïðàâîãî ïîäøèïíèêà äî øåñòåðíè à = 110 ìì, à îò ëåâîãî ïîäøèïíèêà äî øåñòåðíè–b = 80 ìì. Ðàäèàëüíîå äàâëåíèå S = 4 êÍ (Ðèñ. 45).

Ð å ø å í è å. 1. Ïðèíèìàåì ðàñ÷åòíóþ ñõåìó âàëà (Ðèñ.46), êîòîðàÿ áóäåò ïðåäñòàâëÿòü ñîáîé äâóõîïîðíóþ áàëêó ñ ïðèëîæåííîé ê íåé ñîñðåäîòî÷åííîé ñèëîé S.



Ðèñ.46

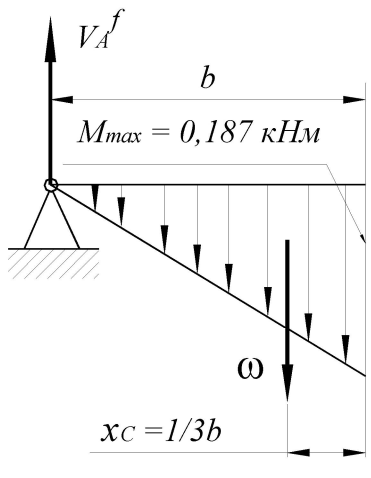
2. Îïðåäåëÿåì âåëè÷èíó îïîðíûõ ðåàêöèé, ñòðîèì ýïþðó èçãèáàþùèõ ìîìåíòîâ è íàãðóæàåì åé ôèêòèâíóþ áàëêó (Ðèñ. 47).



Ðèñ. 47

3. Îïðåäåëÿåì âåëè÷èíó ðåàêöèè:
$SM_B = -V_{\text{À}}^f(b + a) + wx_C = 0$, ñëåäîâàòåëüíî, $V_{\text{À}}^f = w(a + 1)/3(b + a) =$
$= 0{,}5M_{max}(b + a)(a + 1) \ / \ 3(b + a) = 0{,}5\times0{,}187(0{,}11 + 0{,}19) \ / \ 3 = 0{,}009$ êÍ = 9 Í.

4. Îïðåäåëÿåì âåëè÷èíó ôèêòèâíîãî èçãèáàþùåãî ìîìåíòà â ìåñòå ðàñïîëîæåíèÿ øåñòåðíè (Ðèñ. 48).



Ðèñ. 48

$\text{Ì}_f = V_{\text{À}}^f b - wx_C = 9\times0{,}08 - (M_{max}/2)(b^2/3) = 0{,}72 - (1/2)\times187\times0{,}08\times(1/3)\times0{,}08 =$
$= 0{,}52$ Íì^3.

5. Îïðåäåëÿåì âåëè÷èíó ïðîãèáà â ìåñòå óñòàíîâêè øåñòåðíè ïî ôîðìóëå (7.9): $y = \text{Ì}_f / EI$,

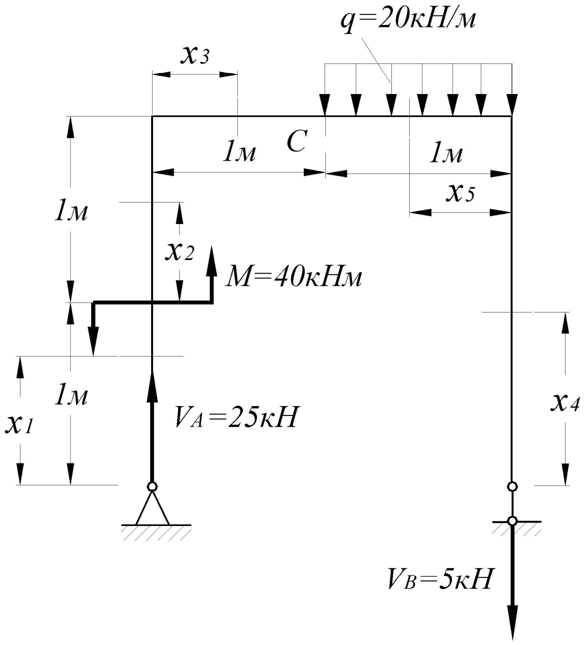
ãäå E– ìîäóëü Þíãà ñòàëè, $2\cdot10^{11}$ Ïà;

I– îñåâîé ìîìåíò èíåðöèè ïîïåðå÷íîãî ñå÷åíèÿ âàëà, $104857\cdot10^{-12}$ ì^4.

Òîãäà $y_{max} = 0{,}52 / (2\cdot10^{11}\cdot104857\cdot10^{-12}) = 0{,}000024$ ì = 0,024 ìì.

Òàê êàê y_{max} = 0,024 ìì < [ó] = 0,1 ìì, *æåñòêîñòü âàëà îáåñïå÷åíà*.

Ïðèìåð 9. Äëÿ çàäàííîé ðàìû (Ðèñ. 50)ñ ïîìîùüþ ìåòîäà Ìàêñâåëëà–Ìîðà îïðåäåëèòü óãîë ïîâîðîòà q_A è ïðîãèá $ó_Ñ$. Æåñòêîñòü êîëîíí è ðèãåëåé ðàìû ïðèíÿòü ðàâíûìè EI.



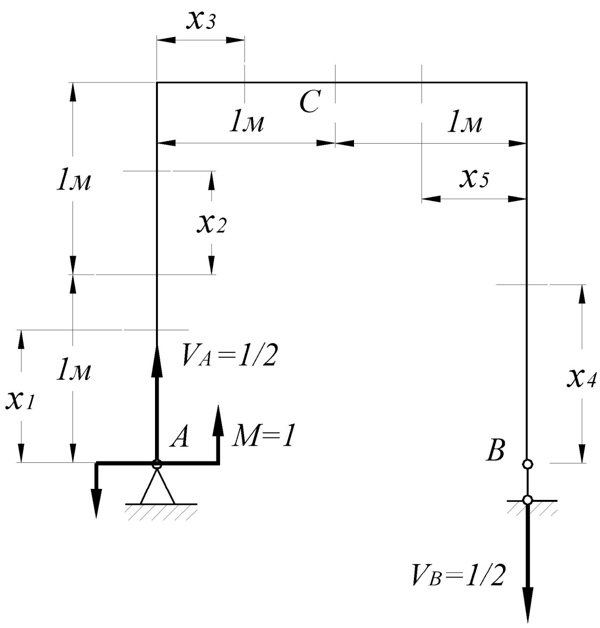
Ðèñ. 50

Ð å ø å í è å. 1. Îïðåäåëÿåì âåëè÷èíó îïîðíûõ ðåàêöèé, ðàçáèâàåì ðàìó íà ñèëîâûå ó÷àñòêè, ïðîâîäèì íà êàæäîì ó÷àñòêå ïîïåðå÷íûå ñå÷åíèÿ (Ðèñ.7.51) è äëÿ êàæäîãî èç íèõ çàïèñûâàåì óðàâíåíèÿ èçãèáàþùèõ ìîìåíòîâ.

M_1 = 0; 0 £ x_1 £ 1 ì; M_2 = –Ì = – 40; 0 £ $õ_2$ £ 1ì;

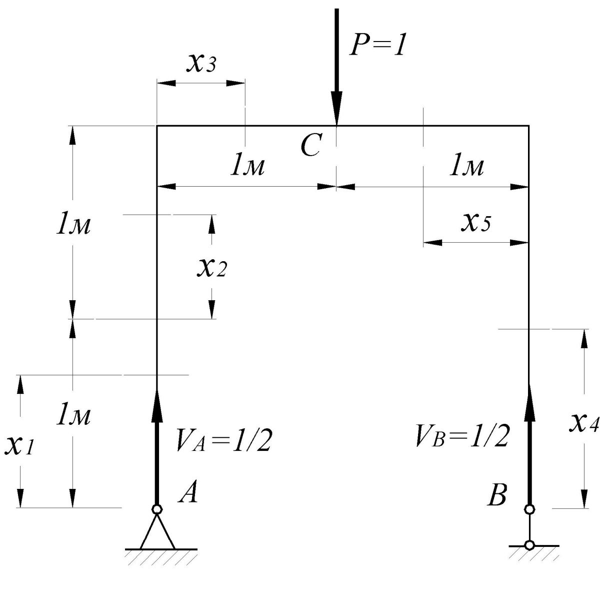
Ìç = –Ì + V_Ax_3 = – 40 + 25$õ_3$; 0 £ $õ_3$ £ 1 ì; $Ì_4$ = 0; 0 £ x_4 £ 1 ì;

$Ì_5$ = –V_Bx_5 – qx_5^2 / 2 = – 5õ – 10$õ^2$; 0 £ $õ_5$ £ 1ì.



Ðèñ. 51

2. Îïðåäåëÿåì óãîë ïîâîðîòà q_A. Ïðèêëàäûâàåì â òî÷êå À ìîìåíò Ì=1 (Ðèñ. 52) è îïðåäåëÿåì âåëè÷èíó îïîðíûõ ðåàêöèé. Â òîé æå ïîñëåäîâàòåëüíîñòè, ÷òî íà äåéñòâèòåëüíîé ðàìå, ïðîâîäèì ïîïåðå÷íûå ñå÷åíèÿ è çàïèñûâàåì óðàâíåíèÿ èçãèáàþùèõ ìîìåíòîâ.



Ðèñ. 52

$M°_1$ = –1; $Ì^0_2$ = –1; $Ì^0_3$ = –1 + x_3 /2; $Ì^0_4$ = 0: $Ì^0_6$ = –x_5 /2.

Èìååì ñëåäóþùèå óðàâíåíèÿ:

M_1 = 0; 0 £ õ £ 1 ì; $M°_1$ = –1; M_2 = – 40; 0 £ õ £ 1 ì; $Ì°_2$ = –1;

Ì$_3$ = – 40 + 25õ; 0 £ õ £ 1 ì; Ì°$_3$ = –1 + x/2; Ì$_4$ =0; 0 £ õ £ l ì; Ì°$_4$ = 0;

Ì$_5$= –5õ – 10õ^2; 0 £ õ £ 1 ì; Ì°$_5$= –x/2.

Âèäíî, ÷òî ïðè îïðåäåëåíèè óãëà ïîâîðîòà q_A ïðîñóììèðóþòñÿ òðè èíòåãðàëà Ìîðà, òàê êàê íà ïåðâîì è ÷åòâåðòîì ó÷àñòêàõ ïðîèçâåäåíèÿ ÌÌ°=0.

$$q_A = (1/EI)\left[\int_0^1 M_2M^0_2dx + \int_0^1 M_3M^0_3dx + \int_0^1 M_5M^0_5dx\right] =$$

$$= (1/EI)\left[\int_0^1 (-40)(-1)dx + \int_0^1 (-40+25x)(-1+x/2)dx + \int_0^1 (-5x-10x^2)(-x/2)dx\right] =$$

$$= (1/EI)(40x + 40x - 25x^2/2 - 20x^2/2 + 12{,}5x^3/3 + 2{,}5x^3/3 + 5x^4/4)\Big|_0^1 =$$

$$= 63{,}75/EI.$$

$q_À$ = 63.75/EI.

3. Îïðåäåëÿåì ïðîãèá ó$_Ñ$. Ïðèêëàäûâàåì â òî÷êå Ñ ñèëó Ð = 1 (Ðèñ. 7.52) è îïðåäåëÿåì âåëè÷èíó îïîðíûõ ðåàêöèé. Â òîé æå ïîñëåäîâàòåëüíîñòè, ÷òî íà äåéñòâèòåëüíîé ðàìå, ïðîâîäèì ïîïåðå÷íûå ñå÷åíèÿ è çàïèñûâàåì óðàâíåíèÿ èçãèáàþùèõ ìîìåíòîâ.

M°$_1$ = 0; Ì°$_2$ = 0; Ì°$_3$ = x$_3$ /2; Ì°$_4$ = 0; M°$_5$ = x$_5$ /2.

Èìååì ñëåäóþùèå óðàâíåíèÿ:

M$_1$ = 0; 0 £ õ £ 1 ì; M°$_1$ = 0; Ì$_2$ = –40; 0 £ õ £ 1 ì; Ì°$_2$ = 0;

Ì$_3$ = –40 + 25õ; 0 £ õ £ l ì; Ì^{0}ç = x/2; Ì$_4$ = 0; 0 £ õ £ 1 Ì; Ì$^0_4$ = 0;

M$_5$ = –5õ – 10õ^2; 0 £ õ £ 1 ì; Ì$^0_5$ = –x/2.

Âèäíî, ÷òî ïðè îïðåäåëåíèè ïðîãèáà ó$_ñ$ ïðîñóììèðóþòñÿ äâà èíòåãðàëà Ìîðà, òàê êàê íà ïåðâîì, âòîðîì è ÷åòâåðòîì ó÷àñòêàõ ïðîèçâåäåíèÿ ÌÌ^0 = 0.

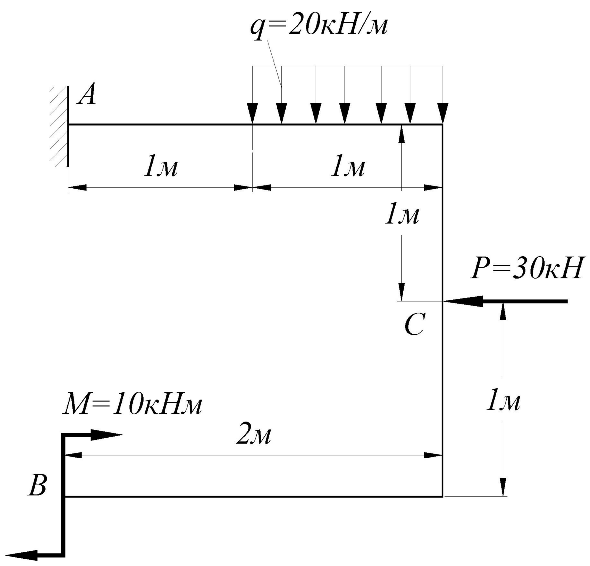
$$y_C = (1/EI)\left[\int_0^1 M_3M^0_3dx + \int_0^1 M_5M^0_5dx\right] =$$

$$= (1/EI)\left[\int_0^1 (-40 + 25x)(x/2)dx + \int_0^1 (-5x - 10x^2)(x/2)\right] =$$

$$= (1/EI)(-20x^2/2 + 12{,}5x^3/3 - 2{,}5x^3/3 - 5x^4/4)\Big|_0^1 = -7{,}92/EI.$$

$$y_C = -7{,}92\ /EI.$$

Ïðèìåð 10. Äëÿ çàäàííîé ðàìû (Ðèñ. 53) ñ ïîìîùüþ ìåòîäà Ìàêñâåëëà–Ìîðà îïðåäåëèòü óãîë ïîâîðîòà q_B è ãîðèçîíòàëüíîå ïåðåìåùåíèå $õ_C$. Æåñòêîñòü êîëîíí è ðèãåëÿ ðàìû ïðèíÿòü ðàâíîé EI.



Ðèñ. 53

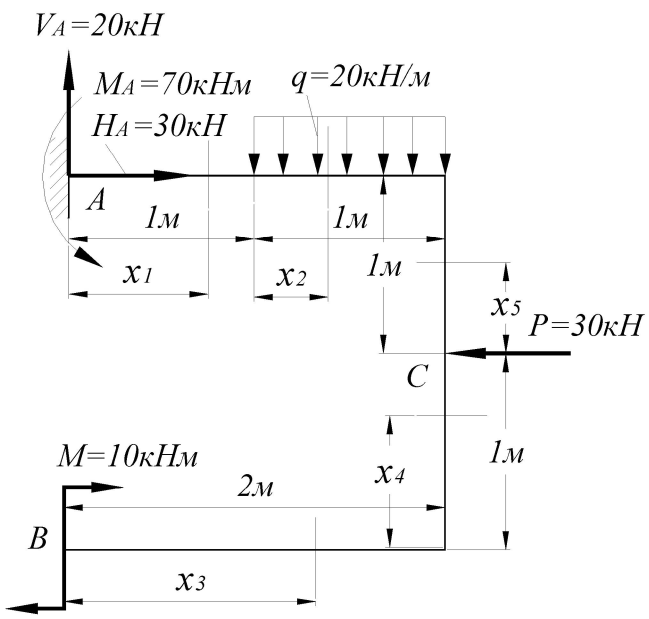
Ð å ø å í è å. 1. Îïðåäåëÿåì âåëè÷èíó îïîðíûõ ðåàêöèé, ðàçáèâàåì ðàìó íà ñèëîâûå ó÷àñòêè, ïðîâîäèì íà êàæäîì ó÷àñòêå ïîïåðå÷íûå ñå÷åíèÿ (Ðèñ. 54) è äëÿ êàæäîãî èç íèõ çàïèñûâàåì óðàâíåíèå èçãèáàþùèõ ìîìåíòîâ.

$M_1 = -Ì_À + V_A x_1 = -70 + 20x_1$; 0 £ x_1 £ 1 ì;

$M_2 = -Ì_À + V_A(x_2 + 1) - qx_2^2 / 2 = -50 + 20x_2 - 10x_2^2$; 0 £ x_2 £ 1 ì;

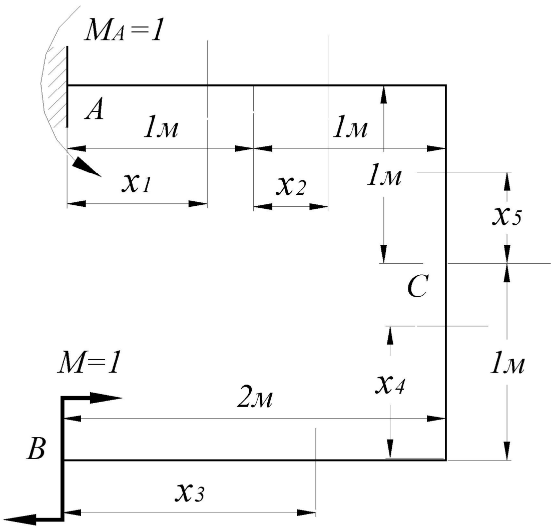
$Ì_3 = -Ì = -10$; 0 £ x_3 £ 1 ì; $Ì_4 = -Ì = -10$; 0 £ x_4 £ 1 ì;

$M_5 = -Ì - Ðõ_5 = -10 - 30õ_5$; 0 5 £ x_5 £ 1 ì.



Ðèñ.54

3. Îïðåäåëÿåì óãîë ïîâîðîòà q$_Â$. Ïðèêëàäûâàåì â òî÷êå Â ìîìåíò Ì = 1 (Ðèñ. 55) è îïðåäåëÿåì âåëè÷èíû îïîðíûõ ðåàêöèé. Â òîé æå ïîñëåäîâàòåëüíîñòè, ÷òî íà äåéñòâèòåëüíîé ðàìå, ïðîâîäèì ïîïåðå÷íûå ñå÷åíèÿ è çàïèñûâàåì óðàâíåíèÿ èçãèáàþùèõ ìîìåíòîâ.



Ðèñ. 55

Ì°$_1$ = –1; Ì$^0_2$= –1; Ì°$_3$ = –1; M$^0_4$= – l; M°$_5$ = –1.

Èìååì ñëåäóþùèå óðàâíåíèÿ:

M$_1$ = –70+20õ; 0 £ x £ 1 ì; M°$_1$ = –1; Ì$_2$ = –50+20õ–10õ^2; 0 £ õ £ 1ì; Ì°$_2$ = –1;

Ì3 = –10; 0 £ õ £ 1ì; Ì°ç = –1; Ì4 = –10; 0 £ x £ 1Ì; Ì°4 = –1;
M5 = –10 – Ç0õ; 0 £ õ £ 1 ì; Ì05 = –1.

1 1 1 1 1

qB = (1/EI)[òM1M01dx + òM2M02dx + òM3M03dx + òM4M04dx + òM5M05dx] =

0 0 0 0 0

1 1 1 1

= (1/EI)[ò(–70+20x)(–1)dx+ò(–50+20x–10x2)(–1)dx+ò(–10)(–1)dx+ò(–10)(–1)dx +

0 0 0 0

1

+ ò(–10 – 30x)(–1)dx] =

0

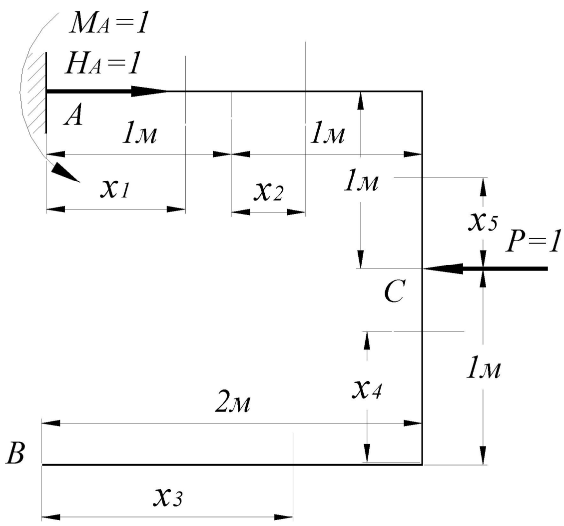
1

= (1/EI)(70x – 25x2 /2 + 50x – 20x2 /2 + 10x3 /3 + 10x + 10x + 10x + 30x2 /2)ô =

0

= 163,33/EI, òîãäà qÂ = 163,33/EI.

4. Îïðåäåëÿåì ãîðèçîíòàëüíîå ïåðåìåùåíèå õñ. Ïðèêëàäûâàåì äëÿ ýòîãî â òî÷êå Ñ ñèëó Ð = 1 (Ðèñ. 56) è îïðåäåëÿåì âåëè÷èíû îïîðíûõ ðåàêöèé. Â òîé æå ïîñëåäîâàòåëüíîñòè, ÷òî íà äåéñòâèòåëüíîé ðàìå, ïðîâîäèì ïîïåðå÷íûå ñå÷åíèÿ è çàïèñûâàåì óðàâíåíèÿ èçãèáàþùèõ ìîìåíòîâ.



Ðèñ. 56

Ì$^0_1$ = 1; Ì$^0_2$ = 1; Ì$^0_3$ = 0; Ì$^0_4$ = 0; Ì$^0_5$ = –x$_5$;

Èìåàì ñëåäóþùèå óðàâíåíèÿ:

M$_1$ = –70 + 20õ; 0 £ x$_1$£ 1 ì; M$^0_1$ = –1; Ì$_2$ = –50 + 20õ – 10õ^2; 0 £x$_2$£ 1 ì; Ì$^0_2$ = –1;

Ì$_3$ = –10; 0 £ õ$_3$ £ 1 ì; Ì$^0_3$ = 0; Ì$_4$ = –10; 0 £ x$_4$ £ 1ì; Ì$^0_4$ = 0;

M$_5$ = –10 – Ç0õ; 0 £ x$_5$ £ 1 ì; Ì$^0_5$ = –õ.

Âèäíî, ÷òî ïðè îïðåäåëåíèè ïåðåìåùåíèÿ õ$_C$ ïðîñóììèðóþòñÿ òîëüêî òðè èíòåãðàëà Ìîðà, òàê êàê íà òðåòüåì è ÷åòâåðòîì ó÷àñòêàõ ïðîèçâåäåíèÿ Ì×Ì^0 = 0.

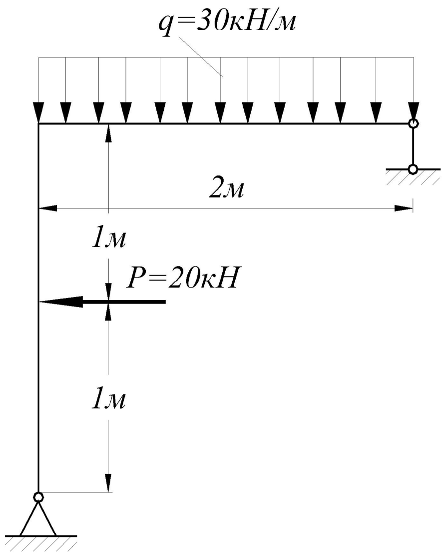
$$x_C = (1/EI)\left(\int_0^1 M_1M^0_1dx + \int_0^1 M_2M^0_2dx + \int_0^1 M_5M^0_5dx\right) =$$

$$= (1/EI)\left[\int_0^1(-70 + 20x)(-1)dx + \int_0^1(-50 + 20x - 10x^2)(-1)dx + \int_0^1(-10 - 30x)(-x)\right] =$$

$$= (1/EI)(70x - 20x^2/2 + 50x - 20x^2/2 + 10x^3/3 + 30x^3/3)\Big|_0^1 = 118{,}33/EI.$$

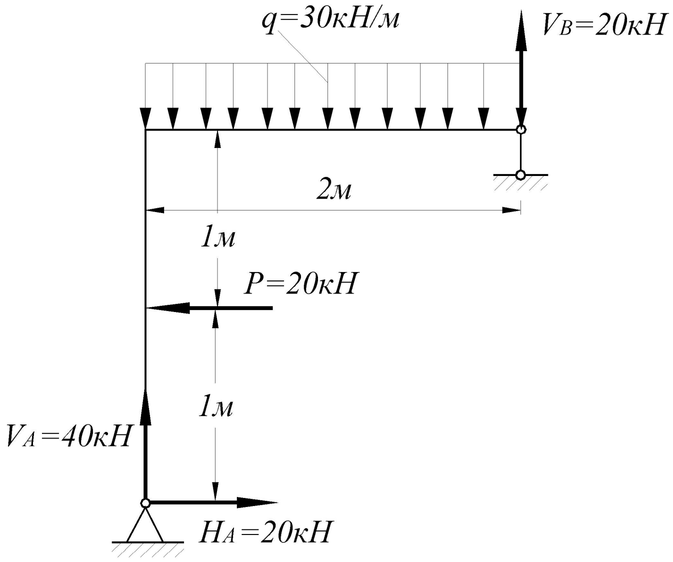
$$x_C = 118{,}33/EI.$$

Ïðèìåð 11. Äëÿ çàäàííîé ðàìû (Ðèñ. 57) îïðåäåëèòü óãîë ïîâîðîòà åå íèæíåãî êîíöà è ãîðèçîíòàëüíîå ïåðåìåùåíèå âåðõíåé îïîðû ìåòîäîì Âåðåùàãèíà. Æåñòêîñòü êîëîííû è ðèãåëÿ ðàìû ïðèíÿòü ðàâíîé EI.



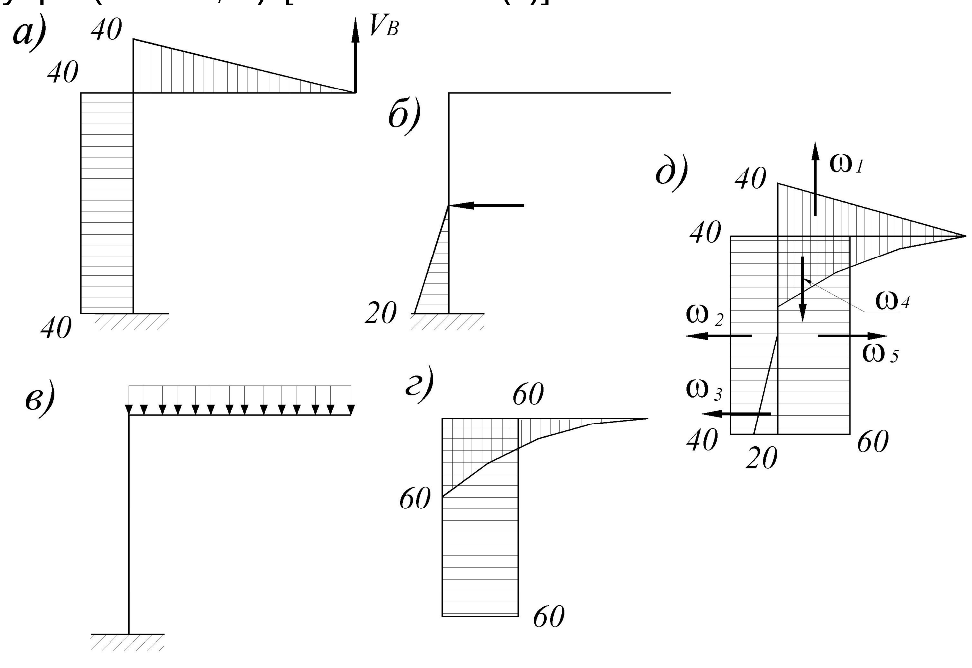
Ðèñ. 57

Ð å ø å í è å. 1. Îïðåäåëÿåì âåëè÷èíû îïîðíûõ ðåàêöèé (Ðèñ. 58).



Ðèñ. 58

2. Ñòðîèì ðàññëîåííóþ ýïþðó èçãèáàþùèõ ìîìåíòîâ â îòäåëüíîñòè îò êàæäîé èç ñèë, ïðèëîæåííûõ ê ðàìå (Ðèñ. 59, à, á, â, ã) è îïðåäåëÿåì ïëîùàäè ïîëó÷åííûõ ýïþð (Ðèñ. 59, ä) [ñì. òàáëèöó (2)].

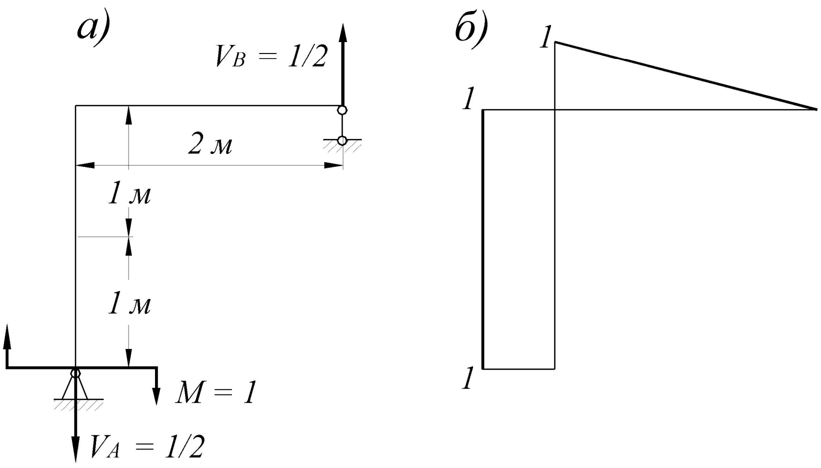


Ðèñ. 59

w_1 = (1/2)40×2 = 40 êÍì^2; w_2 = 40×2 = 80 êÍì^2; w_3 = – (1/3)60×2 = –40 êÍì^2;

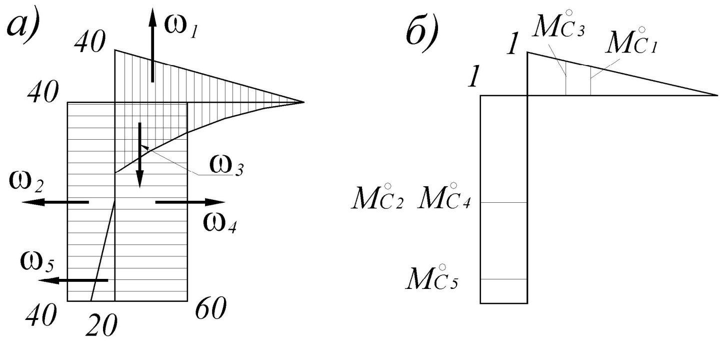
w_4 = –60×2 = –120 êÍì^2; w_5 = (1/2)20×1 = 10 êÍì^2.

3. Îïðåäåëÿåì óãîë ïîâîðîòà íèæíåãî êîíöà ðàìû $q_À$. Ïðèêëàäûâàåì äëÿ ýòîãî â îïîðå ìîìåíò Ì = 1 (Ðèñ. 60, à), îïðåäåëÿåì âåëè÷èíû îïîðíûõ ðåàêöèé è ñòðîèì ýïþðó èçãèáàþùèõ ìîìåíòîâ (Ðèñ. 60, á).



Ðèñ. 60

Îïðåäåëÿåì îðäèíàòû Ì°$_Ñ$ íà ýïþðå îò åäèíè÷íîãî ìîìåíòà (Ðèñ. 61,á), ïîëó÷åííûå ïðè ïðîåöèðîâàíèè íà íåå öåíòðîâ òÿæåñòè ýïþð èçãèáàþùèõ ìîìåíòîâ îò âíåøíèõ ñèë (Ðèñ. 61, à).



Ðèñ. 61

Ïðèìå÷àíèå: íà ðèñóíêå îðäèíàòû Ì$^0_Ñ$ îïðåäåëÿåì èç óñëîâèÿ ïîäîáèÿ òðåóãîëüíèêîâ, ïðè îïðåäåëåíèè ïëîùàäåé ýïþð w è îðäèíàò Ì$^0_Ñ$ ó÷èòûâàåì èõ çíàê.

Ì$^0_{Ñ1}$ = 2/3 ì; Ì$^0_{Ñ2}$ = 1 ì; Ì$^0_{Ñ3}$ = 3/4 ì; Ì$^0_{Ñ4}$ = 1 ì; Ì$^0_{Ñ5}$ = 1 ì.

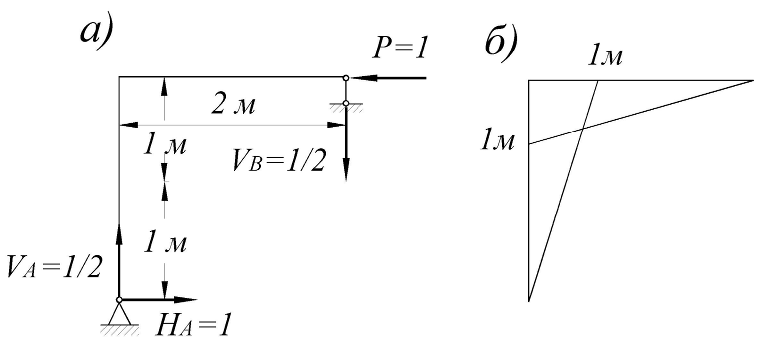
Èìååì ñëåäóþùèå ïëîùàäè è îðäèíàòû:

w_1 = (1/2)40×2 = 40 êÍì^2; Ì$^0_{Ñ1}$ = 2/3 ì; w_2 = 40×2 = 80 êÍì^2; Ì$^0_{Ñ2}$ = 1 ì;
w_3 = –(1/3)60×2 = –40 êÍì^2; Ì$^0_{Ñ3}$ = 3/4 ì; w_4 = –60×2 = –120 êÍì^2; Ì$^0_{Ñ4}$ = 1 ì;
w_5 = (1/2)20×1 = 10 êÍì^2; Ì$^0_{Ñ5}$ = 1 ì. Ïîäñòàâëÿåì â ôîðìóëó (7.21):

$q_À$ = (1/EI)(w_1Ì$^0_{Ñ1}$ + w_2Ì$^0_{Ñ2}$ + w_3Ì$^0_{Ñ3}$ + w_4Ì$^0_{Ñ4}$ + w_5Ì$^0_{Ñ5}$) =

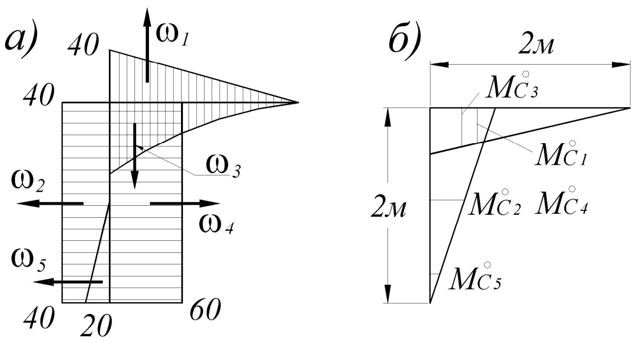
= (1/EI)[40(2/3) + 80 ×1 – 40(3/4) – 120×1 + 10×1] = – 33,33/EI ðàä.

5. Îïðåäåëÿåì ãîðèçîíòàëüíîå ïåðåìåùåíèå õ$_B$ âåðõíåé îïîðû ðàìû. Ïðèêëàäûâàåì äëÿ ýòîãî â òî÷êå Â ñèëó Ð = 1 (Ðèñ. 62, à), îïðåäåëÿåì âåëè÷èíó îïîðíûõ ðåàêöèè è ñòðîèì ýïþðó èçãèáàþùèõ ìîìåíòîâ (Ðèñ.62,á).



Ðèñ. 62

Îïðåäåëÿåì îðäèíàòû M^0_C íà ýïþðå îò åäèíè÷íîé ñèëû (Ðèñ. 63, á), ïîëó÷åííûå ïðè ïðîåöèðîâàíèè íà íåå öåíòðîâ òÿæåñòè ýïþð èçãèáàþùèõ ìîìåíòîâ îò âíåøíèõ ñèë (Ðèñ. 63, à).



Ðèñ. 63

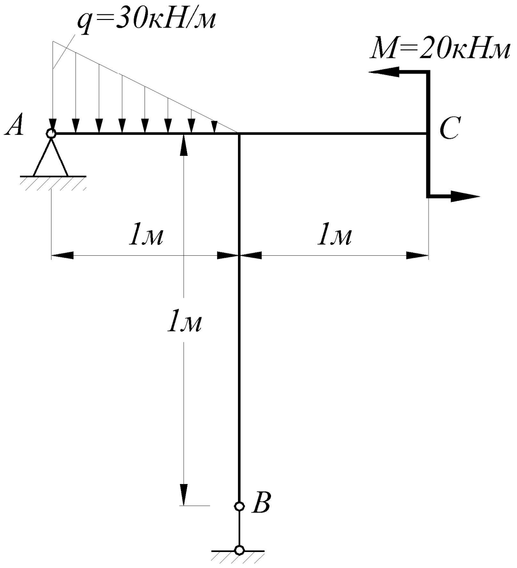
Ì$^0_{Ñ1}$ = – 4/3 ì; Ì$^0_{Ñ2}$ = –1 ì; Ì$^0_{Ñ3}$ = – 3/2 ì; Ì$^0_{Ñ4}$ = –1 ì; Ì$^0_{Ñ5}$ = –1/3 ì.

Èìååì ñëåäóþùèå ïëîùàäè è îðäèíàòû:

w$_1$ = 40 êÍì^2; Ì$^0_{Ñ1}$ = –4/3 ì; w$_2$ = 80 êÍì^2; Ì$^0_{Ñ2}$ = –1 ì;
w$_3$ = –40 êÍì^2; Ì$^0_{Ñ3}$ = –3/2 ì; w$_4$ = – 120 êÍì^2; Ì$^0_{Ñ4}$ = –1 ì;
w$_5$ = 10 êÍì^2; Ì$^0_{Ñ5}$ = –1/3 ì.

x$_B$ = (1/EI)(w$_1$Ì$^0_{Ñ1}$ + w$_2$Ì$^0_{Ñ2}$ + w$_3$Ì$^0_{Ñ3}$ + w$_4$Ì$^0_{Ñ4}$ + w$_5$Ì$^0_{Ñ5}$) =
= (1/EI)[–40(4/3) – 80×1 + 40(3/2) + 120×1 – 10(1/3)] = 43,33/EI ðàä.

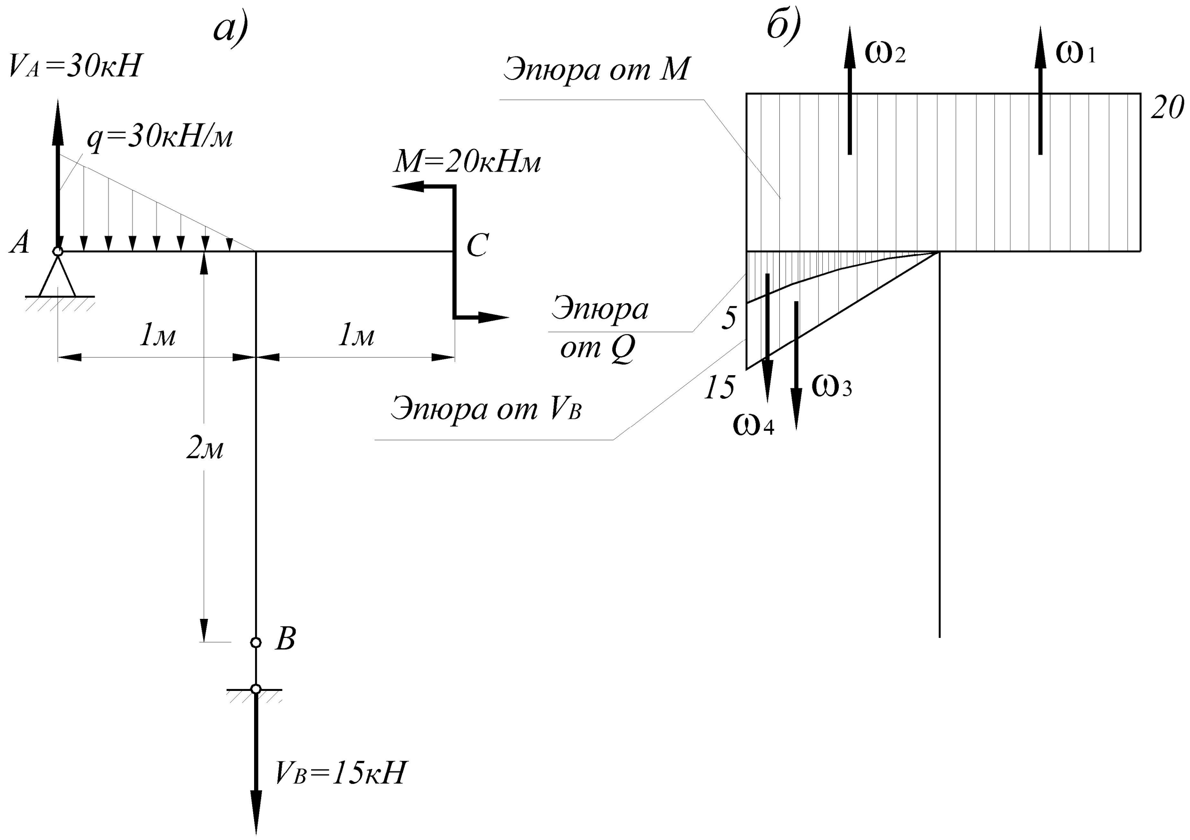
Ïðèìåð 12. Äëÿ çàäàííîé ðàìû (Ðèñ. 64) îïðåäåëèòü óãîë ïîâîðîòà ñå÷åíèÿ, ïðîõîäÿùåãî ÷åðåç ïðàâóþ îïîðó è ïðîãèá ñå÷åíèÿ, ïðîâåäåííîãî ÷åðåç òî÷êó Ñ. Æåñòêîñòü êîëîííû è ðèãåëÿ ðàìû ïðèíÿòü ðàâíîé EI.



Ðèñ. 64

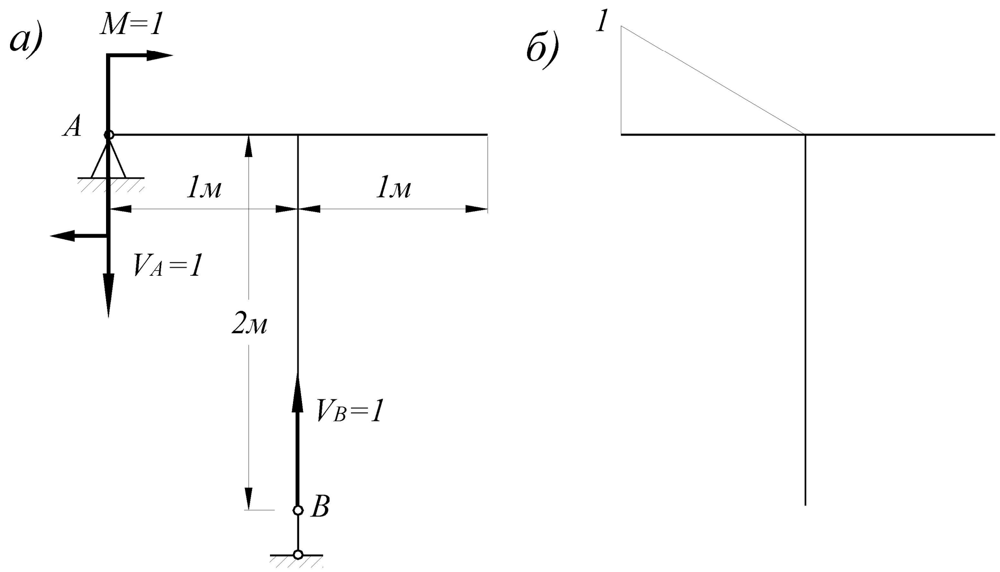
Ð å ø å í è å. 1. Îïðåäåëÿåì âåëè÷èíû îïîðíûõ ðåàêöèé ðàìû (Ðèñ.65, à), ñòðîèì ðàññëîåííóþ ýïþðó èçãèáàþùèõ ìîìåíòîâ â îòäåëüíîñòè îò êàæäîé èç ñèë, ïðèëîæåííûõ ê ðàìå (Ðèñ. 65, á), è îïðåäåëÿåì ïëîùàäè ïîëó÷åííûõ ýïþð.

$w_1 = 20 \times 1 = 20$ кНм^2; $w_2 = 20 \times 1 = 20$ кНм^2; $w_3 = -(1/2)15 \times 1 = -7,5$ кНм^2; $w_4 = -(1/4)5 \times 1 = -5/4$ кНм^2;



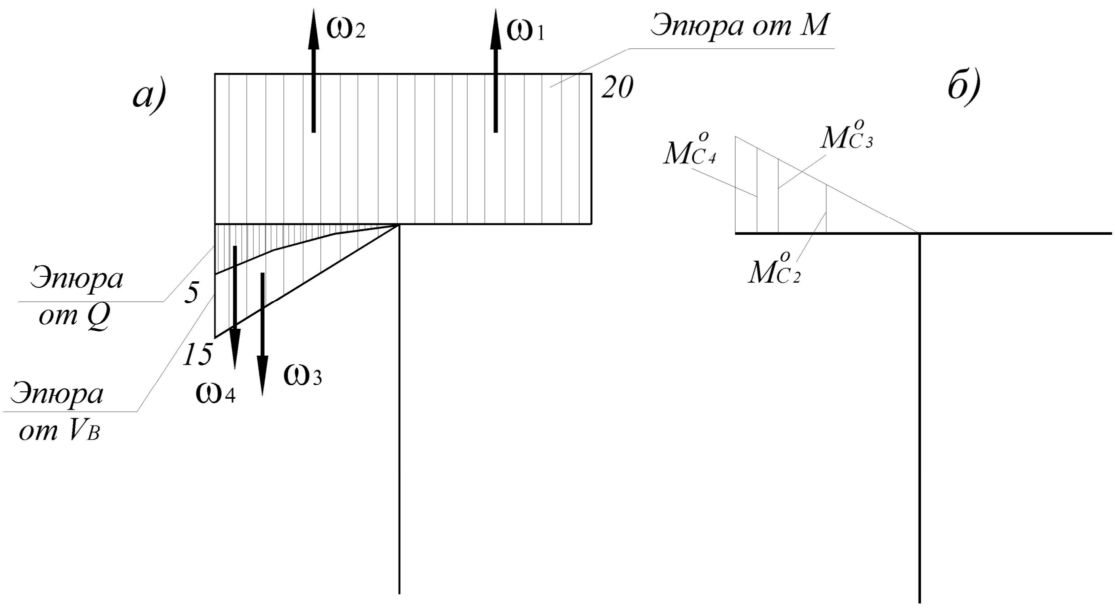
Ðèñ. 65

2. Îïðåäåëÿåì $q_Â$. Ïðèêëàäûâàåì äëÿ ýòîãî â ïðàâîé îïîðå ðàìû ìîìåíò Ì = 1 (Ðèñ. 66, à), îïðåäåëÿåì âåëè÷èíû îïîðíûõ ðåàêöèé è ñòðîèì ýïþðó èçãèáàþùèõ ìîìåíòîâ (Ðèñ. 66, á).



Ðèñ. 66

Îïðåäåëÿåì îðäèíàòû Ì$^{0}_{Ñ}$ íà ýïþðå îò åäèíè÷íîãî ìîìåíòà (Ðèñ. 67,á), ïîëó÷åííûå ïðè ïðîåöèðîâàíèè íà íåå öåíòðîâ òÿæåñòè ýïþð èçãèáàþùèõ ìîìåíòîâ îò âíåøíèõ ñèë (Ðèñ. 67, à).



Ðèñ. 67

Ì$^{0}_{Ñ1}$ = 0; Ì$^{0}_{Ñ2}$ = 0,5 ì; Ì$^{0}_{Ñ3}$ = 2/3 ì; Ì$^{0}_{Ñ4}$ = 4/5 ì.

Èìååì ñëåäóþùèå ïëîùàäè è îðäèíàòû:

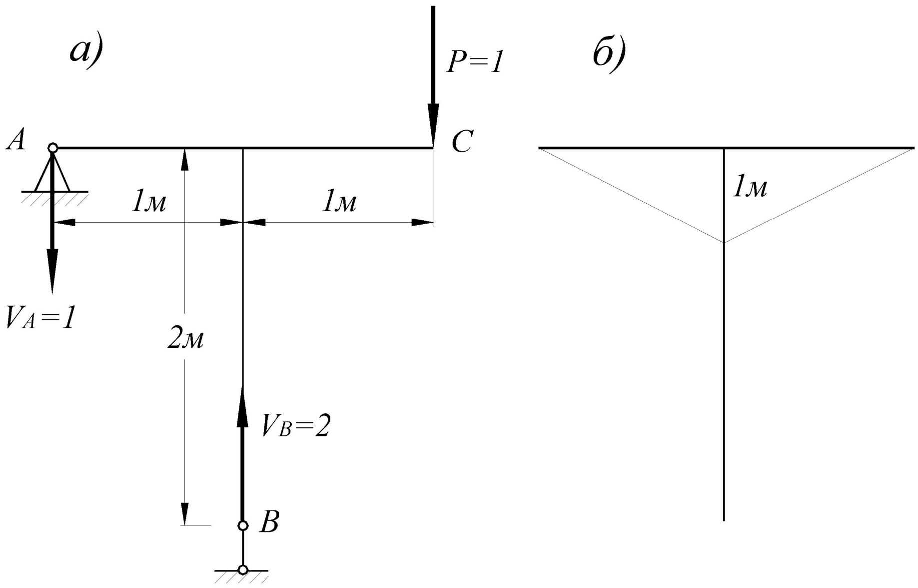
w_1 = 20 êÍì^2; Ì$^{0}_{Ñ1}$ = 0; w_2 = 20 êÍì^2; Ì$^{0}_{Ñ2}$ = 0,5 ì;

w_3 = –7,5 êÍì^2; Ì$^{0}_{Ñ3}$ = 2/3 ì; w_4 = 5/4 êÍì^2; Ì$^{0}_{Ñ4}$ = 4/5 ì;

$q_Â$ = (1/EI)(w_1Ì$^{0}_{Ñ1}$ + w_2Ì$^{0}_{Ñ2}$ + w_3Ì$^{0}_{Ñ3}$ + w_4Ì$^{0}_{Ñ4}$ + w_5Ì$^{0}_{Ñ5}$) =

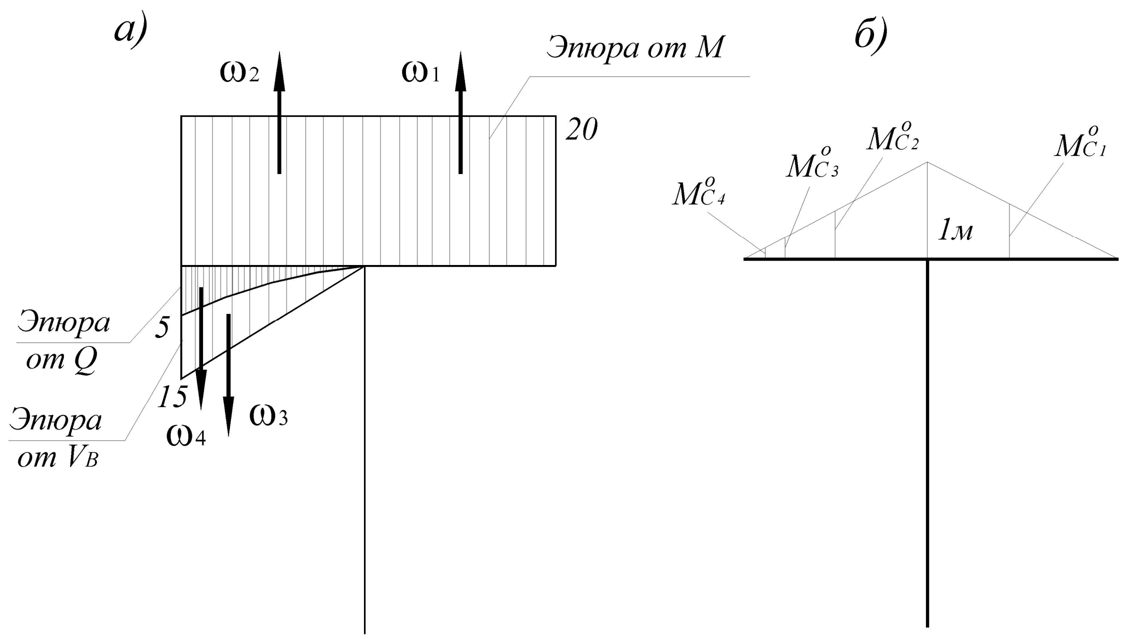
= (1/EI)(20×0,5 – 7,5(2/3) – (5/4)(4/5)) = – 4/EI ðàä.

3. Îïðåäåëÿåì ïîâîðîò ó$_{Ñ}$. Ïðèêëàäûâàåì äëÿ ýòîãî â òî÷êå Ñ ñèëó Ð = 1 (Ðèñ. 68, à), îïðåäåëÿåì âåëè÷èíó îïîðíûõ ðåàêöèé è ñòðîèì ýïþðó èçãèáàþùèõ ìîìåíòîâ (Ðèñ. 68, á).



Ðèñ. 68

Îïðåäåëÿåì îðäèíàòû Ì$^0_Ñ$ íà ýïþðå îò åäèíè÷íîãî ìîìåíòà (Ðèñ. 69, á), ïîëó÷åííûå ïðè ïðîåöèðîâàíèè íà íåå öåíòðîâ òÿæåñòè ýïþð èçãèáàþùèõ ìîìåíòîâ îò âíåøíèõ ñèë (Ðèñ. 69, à).



Ðèñ. 69

Ì$^0_{Ñ1}$ = – 0,5 ì; Ì$^0_{Ñ2}$ = – 0,5 ì; Ì$^0_{Ñ3}$ = –1/3 ì; Ì$^0_{Ñ4}$ = –1/5 ì;

Èìååì ñëåäóþùèå ïëîùàäè è îðäèíàòû:

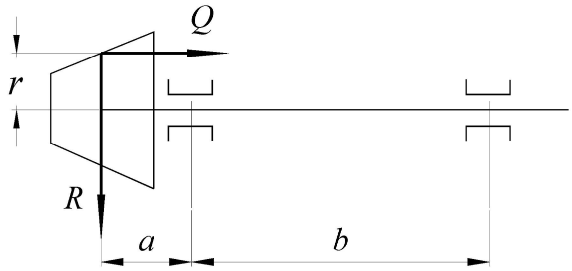
w$_1$ = 20 êÍì^2; Ì$^0_{Ñ1}$ = – 0,5 ì; w$_2$ = 20 êÍì^2; Ì$^0_{Ñ2}$ = – 0,5 ì;

w$_3$ = –7,5 êÍì^2; Ì$^0_{Ñ3}$ = –1/3 ì; w$_4$ = 5/4 êÍì^2; Ì$^0_{Ñ4}$ = –1/5 ì;

q$_Â$ = (1/EI)(w$_1$Ì$^0_{Ñ1}$ + w$_2$Ì$^0_{Ñ2}$ + w$_3$Ì$^0_{Ñ3}$ + w$_4$Ì$^0_{Ñ4}$ + w$_5$Ì$^0_{Ñ5}$) =

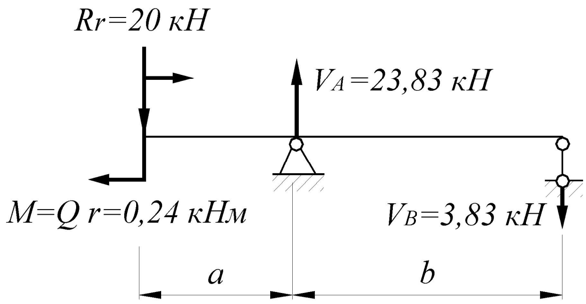
= (1/EI)[–20×0,5 + 7,5×0,5 + 7,5(1/3) + (5/4)(1/5)] = –17,25/EI ðàä.

Ïðèìåð 13. Íàéòè ïðîãèá è óãîë ïîâîðîòà êîíöà âàëà ãëàâíîé ïåðåäà÷è äèàìåòðîì d = 50 ìì (Ðèñ. 70). Ðàçìåðû: à = 35 ìì, b = 120 ìì, ñðåäíèé ðàäèóñ êîíè÷åñêîé øåñòåðíè ã = 120 ìì. Ïðè ðàáîòå ïåðåäà÷è íà âàë ïåðåäàþòñÿ ñëåäóþùèå íàãðóçêè: ðàäèàëüíàÿ R = 20 êÍ, îñåâàÿ Q = 8 êÍ. Ïåðåìåùåíèÿ îïðåäåëèòü ìåòîäîì íà÷àëüíûõ ïàðàìåòðîâ.



Ðèñ. 70

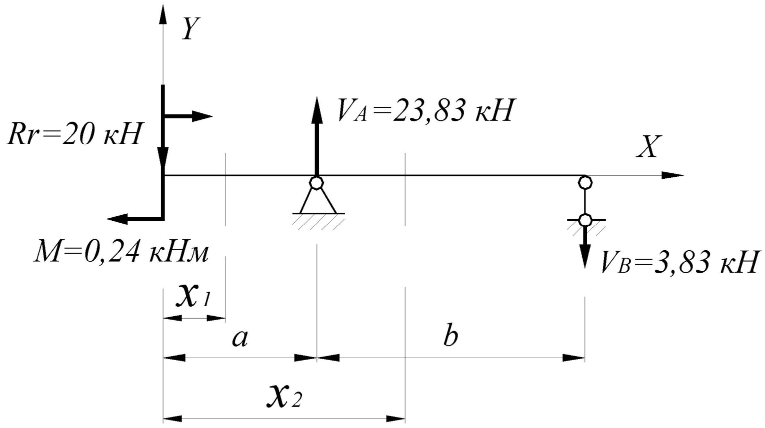
Ð å ø å í è å. 1.Ïðèíèìàåì ðàñ÷åòíóþ ñõåìó âàëà (Ðèñ. 71). Îíà áóäåò ïðåäñòàâëÿòü ñîáîé áàëêó íà äâóõ îïîðàõ ñ ïðèëîæåííûìè íà êîíöå êîíñîëè ñîñðåäîòî÷åííîé ñèëîé R è ìîìåíòîì Ì = Q – r = 8 – 0,03 = 0,24 êÍì.



Ðèñ. 71

2. Îïðåäåëÿåì âåëè÷èíó îïîðíûõ ðåàêöèé:

3. Ïðèíèìàåì íà÷àëî êîîðäèíàò íà ëåâîì êîíöå áàëêè, ïðîâîäèì ïîïåðå÷íûå ñå÷åíèÿ (Ðèñ. 72) è çàïèñûâàåì óðàâíåíèÿ ïðîãèáîâ äëÿ êàæäîãî èç íèõ [ñì. ôîðìóëó (6)].



Ðèñ. 72

$EIy_1=EIy_0+EIq_0x_1+Mx_1^2/2–Rx_1^3/6$; EIy_A(ïðè x_1=0)=EIy_0+EIq_0×0,035+0,0000041= 0;
$EIy_B = EIy_0 + EIq_0x_1 + Mx_2^2/2 – Rx_2^3/6 – V_A(x_2 – 1)^{3-}/6$.
Âèäíî, ÷òî ïðè x_2=a+b, y_B=0, òîãäà EIy_B(ïðè x_2=a+b)=EIy_0+0,155EIq_0–0,0026= 0.

Èìååì äâà óðàâíåíèÿ ñ äâóìÿ íåèçâåñòíûìè:

$EIy_0 + EIq_0$×0,035 + 0,0000041 = 0; (à)

$EIy_0 + EIq_0$×0,155 – 0,0026 = 0. (á)

Âû÷èòàÿ èç óðàâíåíèÿ (à) óðàâíåíèå (á), ïîëó÷èì:
–$EIq_0$0,12 + 0,00267 = 0, òîãäà EIq_0 = 0,022 êÍì^2.

Ïîäñòàâèì çíà÷åíèå EIq_0 = 0,022 êÍì^2 â óðàâíåíèå (á):
$EIy_0 + EIq_0$×0,155 – 0,0026 = 0; EIy_0 = – 0,00078 êÍì^3.

4. Îïðåäåëÿåì ïåðåìåùåíèÿ ïðè ìîäóëå Þíãà ñòàëè Å = 2×10^8 êÏà è îñåâîì ìîìåíòå èíåðöèè îòíîñèòåëüíî íåéòðàëüíîé îñè I =pd^4/32 = 24,07 ñì^4 =
= 61,33×10$^{–8}$ ì^4.

q_0 = 0,022/(2×10^8×61,33×10$^{–8}$) = 0,00018 ðàä = 0,01040^0;

y_0 = – 0,00078/(2×10^8×61,33×10$^{–8}$) = – 0,0000064 ì = 0,0064 ì.